Problems involving relation composition

Prove or disprove: If a relation R, defined as a subset of A x A, is symmetric, then show that R □ R is symmetric as well.

Since R is symmetric, we know that if (a,b)□R, then (b,a)□R. We must prove under this assumption that if (a,b)□R□R, then (b,a)□R□R.

If (a,b)□R□R, by the definition of function composition, we must have an element c□A such that (a,c)□R and (c,b)□R.

But we know R is symmetric. Thus, we can deduce that (c,a)□R and (b,c)□R.

But, if this is the case, by the definition of function composition, we must have (b,a)□R□R, because (b,c)□R and (c,a)□R. This is exactly what we needed to prove. Thus, R□R is symmetric.

Prove: R is transitive □ R□ R □ R.

We must show that if (a,b)□R□R, then (a,b)□R.

First of all, if (a,b)□R□R, then by the definition of function composition, there exists an element c of A such that (a,c)□R and (c,b)□R.

But, we also know that R is transitive. Since we have (a,c)□R and (c,b)□R, we can conclude that (a,b)□R, which is exactly what we were trying to prove. Thus, R□ R □ R.

Let *A* denote an arbitrary set, and let *R* denote a transitive relation over *A*, that is, *R* □ *A* □ *A*, and for all *x, y, z* □ *A*, if (*x, y*) □ *R* and (*y, z*) □ *R* then (*x, z*) □ *R*. Prove that the composition relation $R^2 = R \circ R$ is transitive.

We must show the following:

if (*x, y*) □ *R* o *R and* (*y, z*) □ *R* o *R, then* (*x, z*) □ *R* o *R,*

in order to prove that *R* o *R* is transitive.

Since we have (*x, y*) □ *R* o *R,* □w| (x,w) □ R and (w,y) □ R, by definition of relation composition. Since we also have that R is transitive, it follows that (x,y) □ R.

Since we have (*y, z*) □ *R* o *R,* □v| (y,v) □ R and (v,z) □ R, by definition of relation composition. Since we also have that R is transitive, it follows that (y,z) □ R.

Now, but definition of relation composition, since (x,y) □ R and (y,z) □ R, it follows that (x,z) □ *R* o *R*, which is exactly what we wanted to show.

Problems involving functions

Consider a set *A* = {1, 2, 3}, a set *B* = {*a*, *b*, *c*}, and a set *C* = {*x*, *y*}. Define a relation *R* □ *A* □ *B* as *R* = {(1, *a*), (2, *b*), (2, *c*)}, a relation *S* □ *B* □ *C* as *S* = {(*a*, *x*), (*b*, *y*)}, and a relation *T* □ *A* □ *B* as *T* = {(1, *a*), (2, *c*), (3, *b*)}.

Is R a function? If so, is it an injection?

No, since 2 maps to two different elements in the set B.

Is R^{-1} a function? If so, is it an injection?

Yes, since each of a, b, and c in R^{-1} map to exactly one element. However, R^{-1} is not injective because both b and c map to the same element.

Is S a function? Is S^{-1} a function?

S is not a function because c does not map to any element in the set C. But, S^{-1} is a function since both x and y are mapped to exactly one element in the set B.

Is T a function? If so, is it a surjection?

T is a function. It maps each element in the set A to an unique element is the set B. Since both sets are of an equal size, T is a bijection (and a surjection, of course.)

Is the composed relation $S^{-1} \circ R^{-1}$ a function? If so, is it an injection? Is it a surjection?