Math 132
Midterm Examination 1 Solutions – February 15, 2012

6 multiple choice, 4 long answer. 100 points.

Part I was multiple choice. Only the correct answers are listed here.

1. Let $x_i = \dfrac{3i}{2n} - 1$ for $i = 0, 1, \ldots, n$. These x_i's form a partition of the interval:

   (b) $[-1, \dfrac{1}{2}]$

2. Which of the following is equal to

$$\sum_{i=1}^{19} \frac{1}{i}$$

   (c) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{19}$.

3. Which of the following is an antiderivative of $\frac{1}{1+4x^2}$?

   (d) $\dfrac{1}{2}\tan^{-1}(2x)$

4. $\dfrac{d}{dx}\displaystyle\int_{-\pi}^{x} \sin t^2\, dx$ is equal to

   This problem had an error, and was intended to be $\dfrac{d}{dx}\displaystyle\int_{-\pi}^{x} \sin t^2\, dt$. As this is easy to miss in reading, we accepted the answer for the intended problem:

   (a) $\sin x^2$

   as well as:

   (i) None of the above.

5. If f is a continuous function such that $\int_0^{12} f(t)\,dt = 3$, $\int_2^{12} f(t)\,dt = 4$, and $\int_2^4 f(t)\,dt = 1$, then find $\int_0^4 f(t)\,dt$.

   (e) 0

   (Since $\int_0^4 f(t)\,dt = \int_0^{12} f(t)\,dt - \int_2^{12} f(t)\,dt + \int_2^4 f(t)\,dt$.)

6. Which of the following definite integrals have

$$\sum_{i=1}^{n}(1+\frac{i}{n})^4 \cdot \frac{2}{n}$$

as an associated Riemann sum?

I. $\int_0^2 (1+\frac{x}{2})^4\, dx$

II. $\int_1^3 x^4\, dx$

III. $\int_0^1 2\cdot(1+x)^4\, dx$

(f) I and III only.

Part II was long answer.

1. Integration

(a) (5 points) Evaluate $\displaystyle\int_{-1}^{1} \sin \pi x\, dx$.

Solutions 1: Substitute $u = \pi x$, so that $du = \pi\, dx$, and

$$\int_{-1}^{1} \sin \pi x\, dx = \int_{-\pi}^{\pi} \sin u \frac{1}{\pi} du = \left[-\cos u \cdot \frac{1}{\pi}\right]_{-\pi}^{\pi} = 0.$$

Solution 2: Since sin is an odd function, and $[-\pi, \pi]$ is symmetric around the y-axis, the integral is 0 by symmetry.

(b) (5 points) Evaluate $\displaystyle\int x\sin(x^2)\, dx$.

Substitute $u = x^2$, so that $du = 2x\, dx$, and

$$\int x \sin x^2\, dx = \int \sin u \frac{du}{2} = -\frac{\cos u}{2} + C = -\frac{\cos x^2}{2} + C.$$

(c) (5 points) Evaluate $\displaystyle\int_{\ln 2}^{\ln 3} e^{2x}\sqrt{1+e^x}\, dx$.

Substitute $u = 1 + e^x$, so that $du = e^x\, dx$.

$$\begin{aligned}
\int_{\ln 2}^{\ln 3} e^{2x}\sqrt{1+e^x}\, dx &= \int_3^4 (u-1)\cdot\sqrt{u}\, du = \int_3^4 u^{3/2} - u^{1/2}\, du = \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_3^4 \\
&= (\frac{2}{5}\cdot 4^{5/2} - \frac{2}{3}\cdot 4^{3/2}) - (\frac{2}{5}\cdot 3^{5/2} - \frac{2}{3}\cdot 3^{3/2}) \\
&= \frac{64}{5} - \frac{16}{3} - \frac{18\sqrt{3}}{5} + 2\sqrt{3}.
\end{aligned}$$

(d) (7 points) Solve the initial value problem: If $\frac{d}{dt}f(t) = 8t \cdot (2t^2+1)^4$ and $f(0) = 1$, then find $f(t)$.

We first integrate $8t \cdot (2t^2+1)^4$. Substitute $u = 2t^2 + 1$, so that $du = 4t\,dt$, and we have

$$\int 8t \cdot (2t^2+1)^4\,dt = \int 2u^4\,du = 2\frac{u^5}{5} + C = 2 \cdot \frac{(2t^2+1)^5}{5} + C.$$

We now solve for C. At $t = 0$, we have

$$f(0) = 1 = 2 \cdot \frac{1^5}{5} + C = \frac{2}{5} + C,$$

so $C = \frac{3}{5}$, and $f(t) = 2 \cdot \frac{(2t^2+1)^5}{5} + \frac{3}{5}$.

2. Areas and volumes

(a) (15 points) Find the volume of the object formed by rotating

$$y = \sqrt{\frac{x}{1+x^2}}$$

about the x-axis for $0 \leq x \leq 2$.

The cross-sectional areas perpendicular to the x-axis have area $A(x) = \pi \cdot \frac{x}{1+x^2}$, so the volume is

$$V = \int_0^2 \frac{\pi x}{1+x^2}\,dx.$$

We substitute $u = 1 + x^2$ (so $du = 2x\,dx$) to get

$$= \int_1^5 \frac{\pi}{u}\frac{1}{2}du = \frac{\pi}{2}\left[\ln u\right]_1^5 = \frac{\pi}{2}\ln 5$$

(b) (10 points) Find the area between the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$.

The curves cross when $\sin x = \cos x$, i.e., at $\frac{\pi}{4}$. Plotting test points at e.g. 0 and $\frac{\pi}{2}$, we determine that $\cos x > \sin x$ on $[0, \pi/4]$, and the reverse on $[\pi/4, \pi/2]$. The area is thus

$$\begin{aligned} & \left(\int_0^{\pi/4} \cos x - \sin x\,dx\right) + \left(\int_{\pi/4}^{\pi/2} \sin x - \cos x\,dx\right) \\ = & \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2} \\ = & \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1\right) + \left(-1 - 0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = 2\sqrt{2} - 2. \end{aligned}$$

(The area is symmetric over the line $x = \pi/4$, and this could be used to slightly shorten the calculations.)