MATH 2300 - Calculus III

September 26, 2008

Exam 1

Name: ______________________________

1. (2 points) Find the maximum and minimum values achieved by $\sin(x)$ and $\cos(x)$.

   Let $f(x) = \sin(x)$, $g(x) = \cos(x)$. We wish to find x such that $f'(x) = 0$. This is $\cos(x) = 0$ implying $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. $\sin(\frac{\pi}{2}) = 1 = -\sin(\frac{3\pi}{2}$. Similarly $\cos(0) = 1 = -\cos(\pi)$.

2. (6 points) Show that for any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$. (This is a very important inequality in mathematics and should require minimal calculations).

   By the formula, $a \cdot b = |a||b| \cos \alpha$ but by above, $\cos \alpha \leq 1$, thus $a \cdot b = |a||b|$

3. (8 points) Show that for any two vectors (in $\mathbb{R}^3$), $\mathbf{a}$ and $\mathbf{b}$, $(\mathbf{a} \cdot \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b})$

   Using the geometric formulae, we have $(a \cdot b)^2 = |a|^2|b|^2 \cos^2 \alpha$, and $|a \times b|^2 = |a|^2|b|^2 \sin^2 \alpha$. Their sum is equal to $|a|^2|b|^2 \cos^2 \alpha + |a|^2|b|^2 \sin^2 \alpha = |a|^2|b|^2(\cos^2 \alpha + \sin^2 \alpha) = |a|^2|b|^2$.

4. (8 points) If the vector $\mathbf{a}$ is $\rightarrow$, and the vector $\mathbf{b}$ is $\uparrow$, and $\mathbf{a}$ and $\mathbf{b}$ are both contained in this piece of paper, is $\mathbf{b} \times \mathbf{a}$ pointing into the page (away from you) or out of the page (towards you)? Please state why.

   By the right hand rule, $\mathbf{b} \times \mathbf{a}$ will point into the page (i.e. curl your right hand from b into a. This is a clockwise twist and your thumb points into the page.)

5. For this problem you have a particle that has an acceleration function $\mathbf{a}(t) = \langle 2t, \sin t, \cos t \rangle$.

   (a) (8 points) Find the velocity and position vectors of a particle that has initial velocity, $\mathbf{v}(0) = \langle 1, -1, 0 \rangle$, and initial position $\mathbf{r}(0) = \langle 0, 1, 0 \rangle$.

   $\mathbf{v}(t) = \int a(t) + c = \langle t^2, -\cos t, \sin t \rangle + c$. Substituting $t = 0$ into this equation, we get $v(0) = \langle 1, -1, 0 \rangle = \langle 0, -1, 0 \rangle + c$. Thus $c = \langle 1, 0, 0 \rangle$, and $v(t) = \langle t^2 + 1, -\cos t, \sin t \rangle$.

   $\mathbf{r}(t) = \int v(t) + c = \langle t^3/3 + t, -\sin t, -\cos t \rangle + c$. Substituting $t = 0$ into this equation we get $\mathbf{r}(0) = \langle 0, 1, 0 \rangle = \langle 0, 0, -1 \rangle + c$. Thus $c = \langle 0, 1, 1 \rangle$, and $\mathbf{r}(t) = \langle t^3/3 + t, -\sin t + t + 1, -\cos t + 1 \rangle$.

   (b) (8 points) Use your results for $\mathbf{r}(t)$ to find the equations of the unit tangent vector $\mathbf{T}(t)$, unit normal vector $\mathbf{N}(t)$, and binormal vector $\mathbf{B}(t)$.

   $\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{v(t)}{|v(t)|} = \frac{\langle t^2+1, -\cos t, \sin t \rangle}{|\langle t^2+1, -\cos t+1, \sin t \rangle|} = \frac{\langle t^2+1, -\cos t, \sin t \rangle}{\sqrt{(t^2+1)^2 + (-\cos t)^2 + (\sin t)^2}} = \frac{\langle t^2+1, -\cos t, \sin t \rangle}{(t^2+1)} = \langle 1, \frac{-\cos t}{t^2+1}, \frac{\sin t}{t^2+1} \rangle$

   $\mathbf{N}(t) =$

   (c) (8 points) Find the equation of the plane (that changes with time) containing $\mathbf{N}(t)$ and $\mathbf{T}(t)$.

6. For this problem we will work with a plane given by $x - y + 2z = 8$.

   (a) (6 points) Find the point at which the line $\mathbf{r}(t) = \langle 2 - t, 2 + t, 5t \rangle$ intersects the plane.

   Substitute $\mathbf{r}(t)$ into the planar equation: $2 - t - (2 + t) + 2(5t) = 8$ and solve for $t = 1$, plug $t = 1$ into $r(t)$ to get $r(1) = \langle 1, 3, 5 \rangle$

   (b) (8 points) Find the distance between the point $p_1 = \langle -1, 1, 0 \rangle$ and the plane.

   The distance is given by the length of the projection of the vector from a point on the plane to p_1 onto the normal vector. The normal vector $\mathbf{n} = \langle 1, -1, 2 \rangle$ which comes directly from the plane equation. We choose an easy point on the plane: $p_0 = \langle -1, 1, 5 \rangle$, find the vector from p_0 to p_1: $\mathbf{v} = p_1 - p_0 = \langle 0, 0, -5 \rangle$. Finally, the length of the projection is $\mathbf{n} \cdot \mathbf{v}/|\mathbf{n}| = (0 + 0 + -5 * 2)/\sqrt{1 + 1 + 2^2} = 10/\sqrt{6}$.

(c) (8 points) Find the angle between the above plane and the one given by $x + y + z = 1$. You may leave your answer in terms of arccos or arcsin. The normal derivatives are $n_1 = \langle 1, -1, 2\rangle$ and $n_2 = \langle 1, 1, 1\rangle$, and the angle is given by $\cos\alpha = \frac{n_1 \cdot n_2}{|n_1||n_2} = (1 - 1 + 2)\sqrt{6}\sqrt{3} = \frac{2}{\sqrt{18}}$.

7. For this problem let $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 2 + 2t\rangle$.

   (a) (7 points) Find the curvature $\kappa(t)$ as a function of t of the space curve given by $\mathbf{r}(t)$.
   $\kappa(t) = \frac{|T'(t)|}{|r'(t)|}$. $T(t) = \frac{r'(t)}{|r'(t)|} =$

   (b) (7 points) Find the arc length of $\mathbf{r}(t)$ for $0 \le t \le \pi$. arclength$= \int_0^\pi |r'(t)|$

8. For this problem, $f(x, y) = \sqrt{25 - y^2 - x^2}$.

   (a) (2 points) Sketch the domain of $f(x, y)$. $D = \{x, y \in \mathbb{R}^2 : 25 - y^2 - x^2 > 0\}$ this simplifies to $D = \{x, y \in \mathbb{R}^2 : 25 > y^2 + x^2\}$. This is the set of all x, y inside the circle of radius 5.

   (b) (2 points) What is the range of $f(x, y)$. The range of $f(x, y)$ is found by determining the maximum and minimum values $f(x, y)$ can attain: $\max f = 3$ and $\min f = 0$.

   (c) (4 points) What is $f_x(x, y)$. Taking the partial derivative with respect to x, we set all other variables equal to a constant and differentiate as usual: $g(x) = f(x, y) \to g'(x) = f_x(x, y) = \frac{-2x}{2\sqrt{25-y^2-x^2}}$

   (d) (8 points) Sketch the level curves at $f(x, y) = k$ for $k = 0, 1, 2, 3$. Solving the equation $\sqrt{9 - y^2 - x^2} = k$ we can square both sides to get $9 - y^2 - x^2 = k^2$ or $y^2 + x^2 = 9 - k^2$. This is simply a general equation of a circle of radius $\sqrt{9 - k^2}$, or concentric circles of radii 5,3,0.