Math 132
Midterm Examination 2 Solutions – March 26, 2012

6 multiple choice, 4 long answer. 100 points.

Part I was multiple choice. Only the correct answers are listed here.

1. Find the Trapezoid Rule approximation using 4 subintervals of

$$\int_{-1}^{1} x^2\,dx.$$

   (f) 3/4

2. Find the Simpson's Rule approximation using 4 subintervals of

$$\int_{-1}^{1} x^2\,dx.$$

   (e) 2/3

3. Consider the system consisting of 3 point masses:
   10 kg at $(3,-1)$
   20 kg at $(2,10)$
   100 kg at $(1,0)$
   The center of mass is:

   (g) $(\frac{17}{13}, \frac{19}{13})$

4. Simpson's Rule applied to the integral $\int_1^e \frac{1}{x}\,dx$ with $n = 20$ will be closest to:

   (k) 1
   (Since $\frac{1}{x} \le 1$ on $[1, e]$, and then the error bound of $\frac{1\cdot(e-1)^5}{20^4}$ is quite small.)

5. Find the average value of $\sin x$ over the interval $[0, \pi]$.

   (d) $2/\pi$

6. The decay of a certain radioactive isotope of the element rabbitonium is governed by the differential equation $y' = -ky$. At $t = 0$ you have 300 mg of radioactive rabbitonium. At $t = 45$ minutes, you are left with only 100 mg of radioactive rabbitonium.
Then k is ______ per minute.

   (f) $\ln 3/45$.

Part II was long answer.

1. Differential equations

   (a) (8 points) Solve the differential equation $y' = x + xy$ subject to the initial condition $y(0) = 5$.

   Separating the equation, we have

$$\frac{y'}{1+y} = x$$

   hence

$$\begin{aligned} \int \frac{1}{1+y}\,dy &= \int x\,dx \\ \ln ||1+y| &= \frac{x^2}{2} + C \\ 1+y &= Ae^{x^2/2} \\ y &= Ae^{x^2/2} - 1. \end{aligned}$$

   The initial condition $y(0) = 5 = Ae^0 - 1$ gives that $A = 6$, so

$$y = 6e^{x^2/2} - 1.$$

   (b) (8 points) At time $t = 0$, there is 1000 liters of water in a tank, with 80 kg of salt dissolved in it. Distilled water flows into the tank at 10 L/min, and water flows out of the tank at the same rate. The tank is continually stirred, and the salt is kept mixed evenly through the tank.
   Set up a differential equation (you needn't solve it) for the mass of salt in the tank at time t. (Your answer should be of the form $y' =$ ____.)

   Inflow of salt = 0,
   outflow of salt = (amount of salt in tank/1000) · 10,
   so if y = amount of salt in tank, then

$$y' = -\frac{y \cdot 10}{1000}.$$

   The initial condition is $y(0) = 80$.

2. Arc lengths and approximate integration

(a) (6 points) Set up a definite integral representing the length of the curve $y = x^3$ between $x = 0$ and $x = 4$.

$$\int_0^4 \sqrt{1 + (3x^2)^2}\, dx.$$

(b) (10 points) The first several derivatives of $f(x) = \sqrt{1 + x^2}$ are as follows:

$$f'(x) = \frac{x}{\sqrt{1+x^2}}, \qquad f''(x) = \frac{1}{(1+x^2)^{3/2}}, \qquad f^{(3)}(x) = \frac{-3x}{(1+x^2)^{5/2}},$$
$$f^{(4)}(x) = \frac{12x^2 - 3}{(1+x^2)^{7/2}}, \qquad f^{(5)}(x) = \frac{45x - 60x^3}{(x^2+1)^{9/2}}.$$

Find (with justification) an n such that the Simpson's Rule approximation S_n for $\int_{-1}^4 \sqrt{1+x^2}\, dx$ has error at most 0.001.

The main step in this problem is finding an upper bound for $f^{(4)}$.
Approach 1 to bounding $f^{(4)}$: (triangle inequality)
We have that

$$|f^{(4)}(x)| = \frac{|12x^2 - 3|}{|1+x^2|^{7/2}} \leq \frac{12\,|x^2| + 3}{|1+x^2|^{7/2}}.$$

The top is $\leq 12 \cdot 4^2 + 3$ on $[-1, 4]$, and the bottom is ≥ 1 everywhere, hence $\left|f^{(4)}(x)\right| \leq \frac{12 \cdot 16 + 3}{1} = 195$.
Approach 2 to bounding $f^{(4)}$: (take another derivative)
The 5th derivative is continuous on $[1, 4]$, and has roots at 0 and $\pm\frac{\sqrt{3}}{2}$. We approximate these points and the endpoints, using the triangle inequality to simplify:

$$\begin{aligned}
\left|f^{(4)}(-1)\right| &= \frac{12 - 3}{(1+1)^{7/2}} \leq \frac{9}{2^{6/2}} = \frac{9}{8} \\
\left|f^{(4)}(-\frac{\sqrt{3}}{2})\right| &= \frac{12 \cdot \frac{3}{4} - 3}{(1 + \frac{3}{4})^{7/2}} = \frac{6}{(\frac{7}{4})^{7/2}} \leq \frac{6}{(\frac{3}{2})^{6/2}} = \frac{16}{9} \\
\left|f^{(4)}(0)\right| &= \frac{3}{1^{7/2}} = 3 \\
\left|f^{(4)}(\frac{\sqrt{3}}{2})\right| & \quad \text{is the same as } \left|f^{(4)}(-\frac{\sqrt{3}}{2})\right| \\
\left|f^{(4)}(4)|\right| &= \frac{12 \cdot 16 - 3}{(1+16)^{7/2}} \leq \frac{189}{16^{7/2}} = \frac{189}{4^7} \leq 1
\end{aligned}$$

Since the max of $|f^{(4)}(x)|$ on $[-1, 4]$ occurs at one of the above points (as it is clearly zero at the points where it fails to be differentiable), we have that

$$\left|f^{(4)}(x)\right| \leq 3$$