

Math 132

Midterm Examination 2 Solutions – March 26, 2012

6 multiple choice, 4 long answer. 100 points.

Part I was multiple choice. Only the correct answers are listed here.

1. Find the Trapezoid Rule approximation using 4 subintervals of

$$\int_{-1}^1 x^2 dx.$$

(f) $3/4$

2. Find the Simpson's Rule approximation using 4 subintervals of

$$\int_{-1}^1 x^2 dx.$$

(e) $2/3$

3. Consider the system consisting of 3 point masses:

10 kg at $(3, -1)$

20 kg at $(2, 10)$

100 kg at $(1, 0)$

The center of mass is:

(g) $(\frac{17}{13}, \frac{19}{13})$

4. Simpson's Rule applied to the integral $\int_1^e \frac{1}{x} dx$ with $n = 20$ will be closest to:

(k) 1

(Since $\frac{1}{x} \leq 1$ on $[1, e]$, and then the error bound of $\frac{1 \cdot (e-1)^5}{20^4}$ is quite small.)

5. Find the average value of $\sin x$ over the interval $[0, \pi]$.

(d) $2/\pi$

6. The decay of a certain radioactive isotope of the element rabbitonium is governed by the differential equation $y' = -ky$. At $t = 0$ you have 300 mg of radioactive rabbitonium. At $t = 45$ minutes, you are left with only 100 mg of radioactive rabbitonium. Then k is _____ per minute.

(f) $\ln 3/45$.

Part II was long answer.

1. Differential equations

- (a) (8 points) Solve the differential equation $y' = x + xy$ subject to the initial condition $y(0) = 5$.

Separating the equation, we have

$$\frac{y'}{1+y} = x$$

hence

$$\begin{aligned} \int \frac{1}{1+y} dy &= \int x dx \\ \ln ||1+y| &= \frac{x^2}{2} + C \\ 1+y &= Ae^{x^2/2} \\ y &= Ae^{x^2/2} - 1. \end{aligned}$$

The initial condition $y(0) = 5 = Ae^0 - 1$ gives that $A = 6$, so

$$y = 6e^{x^2/2} - 1.$$

- (b) (8 points) At time $t = 0$, there is 1000 liters of water in a tank, with 80 kg of salt dissolved in it. Distilled water flows into the tank at 10 L/min, and water flows out of the tank at the same rate. The tank is continually stirred, and the salt is kept mixed evenly through the tank.

Set up a differential equation (you needn't solve it) for the mass of salt in the tank at time t . (Your answer should be of the form $y' = \underline{\hspace{1cm}}$.)

Inflow of salt = 0,

outflow of salt = (amount of salt in tank/1000) $\cdot$ 10,

so if y = amount of salt in tank, then

$$y' = -\frac{y \cdot 10}{1000}.$$

The initial condition is $y(0) = 80$.

2. Arc lengths and approximate integration

- (a) (6 points) Set up a definite integral representing the length of the curve $y = x^3$ between $x = 0$ and $x = 4$.

$$\int_0^4 \sqrt{1 + (3x^2)^2} dx.$$

- (b) (10 points) The first several derivatives of $f(x) = \sqrt{1 + x^2}$ are as follows:

$$\begin{aligned} f'(x) &= \frac{x}{\sqrt{1 + x^2}}, & f''(x) &= \frac{1}{(1 + x^2)^{3/2}}, & f^{(3)}(x) &= \frac{-3x}{(1 + x^2)^{5/2}}, \\ f^{(4)}(x) &= \frac{12x^2 - 3}{(1 + x^2)^{7/2}}, & f^{(5)}(x) &= \frac{45x - 60x^3}{(x^2 + 1)^{9/2}}. \end{aligned}$$

Find (with justification) an n such that the Simpson's Rule approximation S_n for $\int_{-1}^4 \sqrt{1 + x^2} dx$ has error at most 0.001.

The main step in this problem is finding an upper bound for $f^{(4)}$.

Approach 1 to bounding $f^{(4)}$: (triangle inequality)

We have that

$$|f^{(4)}(x)| = \frac{|12x^2 - 3|}{|1 + x^2|^{7/2}} \leq \frac{12|x^2| + 3}{|1 + x^2|^{7/2}}.$$

The top is $\leq 12 \cdot 4^2 + 3$ on $[-1, 4]$, and the bottom is ≥ 1 everywhere, hence $|f^{(4)}(x)| \leq \frac{12 \cdot 16 + 3}{1} = 195$.

Approach 2 to bounding $f^{(4)}$: (take another derivative)

The 5th derivative is continuous on $[1, 4]$, and has roots at 0 and $\pm \frac{\sqrt{3}}{2}$. We approximate these points and the endpoints, using the triangle inequality to simplify:

$$\begin{aligned} |f^{(4)}(-1)| &= \frac{12 - 3}{(1 + 1)^{7/2}} \leq \frac{9}{2^{6/2}} = \frac{9}{8} \\ \left| f^{(4)}\left(-\frac{\sqrt{3}}{2}\right) \right| &= \frac{12 \cdot \frac{3}{4} - 3}{(1 + \frac{3}{4})^{7/2}} = \frac{6}{(\frac{7}{4})^{7/2}} \leq \frac{6}{(\frac{3}{2})^{6/2}} = \frac{16}{9} \\ |f^{(4)}(0)| &= \frac{3}{1^{7/2}} = 3 \\ \left| f^{(4)}\left(\frac{\sqrt{3}}{2}\right) \right| &\text{ is the same as } \left| f^{(4)}\left(-\frac{\sqrt{3}}{2}\right) \right| \\ |f^{(4)}(4)| &= \frac{12 \cdot 16 - 3}{(1 + 16)^{7/2}} \leq \frac{189}{16^{7/2}} = \frac{189}{4^7} \leq 1 \end{aligned}$$

Since the max of $|f^{(4)}(x)|$ on $[-1, 4]$ occurs at one of the above points (as it is clearly zero at the points where it fails to be differentiable), we have that

$$|f^{(4)}(x)| \leq 3$$