MATH 2300 - Calculus III

October 24, 2008

Exam 2

Name: ______________________________

1. The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of $\frac{2}{R}$ moles of an ideal gas are related by the equation $PV = 2T$. Find the rate at which the pressure is changing when the temperature is 100 K and increasing at a rate of 0.5 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

   Solution: Differentiate with respect to t using the chain rule: $\frac{\partial P}{\partial t}V + \frac{\partial V}{\partial t}P = 2\frac{\partial T}{\partial t}$, solve for $\frac{\partial P}{\partial t} = \frac{1}{V}\left(\frac{\partial T}{\partial t} - \frac{\partial V}{\partial t}P\right)$, and then note that $P = 2T/V$, so $\frac{\partial P}{\partial t} = \frac{1}{V}\left(\frac{\partial T}{\partial t} - \frac{\partial V}{\partial t}\frac{2T}{V}\right)$. Plugging in the values from the problem gives

$$\frac{\partial P}{\partial t} = \frac{1}{100}\left(.5 - .2\frac{2 \times 100}{100}\right) = .1/100 = .001.$$

2. Find $y'(x)$ if $x^3 + y^3 = 6xy$.

   Solution: Differentiate implicitly with respect to x: $3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$ and solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}.$$

3. Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point (0,1) has the value 1.

   Solution: Let $u = (u_1, u_2)$ be the direction, and then we have to solve $\nabla f(0, 1) \cdot u = 1$ and $u \cdot u = 1$ (u is a unit vector) for u_1 and u_2. $\nabla f(x, y) = \langle -y^2e^{-xy}, e^{-xy} - xye^{-xy}\rangle$. Plugging in $(x, y) = (0, 1)$ we get $\nabla f(0, 1) = \langle -1, 1\rangle$. We then need to solve $-u_1 + u_2 = 1$ and $u_1^2 + u_2^2 = 1$. $u_2 = 1 + u_1$ and so $u_2^2 = 1 + 2u_1 + u_1^2$ so $2u_1^2 + 2u_1 = 0$ and thus $u_1 = 0$ or $u_1 = -1$. Thus the directions are $\langle 0, 1\rangle$ or $\langle -1, 0\rangle$.

4. A package in the shape of a rectangular box can be mailed by the USPS if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 120 inches. Find the dimensions of the package with the largest volume that can be mailed.

   Solution: The function to maximize is $f(l, w, h) = lwh$ and the constraint is $g(l, w, h) = l + 2w + 2h = 120$. The method of Lagrange says that we solve $\nabla f = \lambda\nabla g$ or $\langle wh, lh, lw\rangle = \langle\lambda, 2\lambda, 2\lambda\rangle$. Thus we have four equations and four unknowns. $dw = lw$ gives $d = l$ and then $dw = \frac{1}{2}dl$ gives $2w = l$ Thus $2w + 2w + 2w = 6w = 120$ and $w = 20$. So the maximum is achieved at $(l, w, d) = (40, 20, 20)$.

5. Find the volume of the solid enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$ and $y = 4$. (You may use double or triple integration).

   Solution: $z = x^2$ is the surface above the xy plane so we integrate $\int\int_D x^2 dA$ where D is the region defined in the xy plane. This is a parabola bounded by $y = 4$ and $y = x^2$. Since $x = 0$ gives $y = 0$, we integrate from $y = x^2$ to $y = 4$. Finally, we integrate in x from $x = -2$ to $x = 2$ since $y = x^2 = 4$ is the upper bound. So the integral is

$$\int_{-2}^{2}\int_{x^2}^{4} x^2 dy dx = \int_{-2}^{2} 4x^2 - x^4 dx = 4/3x^3 - x^5/5|_{-2}^{2} = 128/15$$

6. Show that

$$0 \leq \iint_T \sin^4(x + y) dA \leq 1$$

where T is the triangle enclosed by the lines $y = 0$, $y = 2x$ and $x = 1$.

Solution: $0 < \sin^4(x+y) < 1$ for all x and y. Thus

$$0A(T) \leq \iint_T \sin^4(x+y)dA \leq 1A(T).$$

So we calculate the area of T: this is simply a triangle of base 1 and height 2. (or integrate $y = 2x$ from $x = 0$ to $x = 1$, and the area is 1. Therefore

$$0 \leq \iint_T \sin^4(x+y)dA \leq 1.$$

7. Set up the following integral in polar coordinates but do not evaluate it:

$$\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2 y \; dxdy.$$

Solution: The region of integration is the segment of the disk of radius a from $\pi/2$ to π (note that the limits of integration go from $y = 0$ to $y = a$. Therefore the change of coordinates to polar gives

$$\int_{\pi/2}^{\pi} \int_0^a r^2 \cos^2\theta r \sin\theta r dr d\theta = \int_{\pi/2}^{\pi} \int_0^a r^4 \cos^2\theta \sin\theta dr d\theta$$

8. Suppose X and Y are random variables with joint density function

$$f(x,y) = \begin{cases} 0.1e^{-(0.5x+0.2y)} & \text{if } x \geq 0, \;\; y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that f is indeed a joint density function.

(b) Find the probability $P(X \leq 2, Y \leq 5)$. You may leave your answer in terms of exponents (e.g. $e^a - e^b$).

(c) Find the expected value of X (hint: remember integration by parts!).

Solution: For convenience we rewrite f as $f(x,y) = .1e^{-.5x}e^{-.2y}$. f is a JDF if $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \int_0^{\infty}\int_0^{\infty} .1e^{-.5x}e^{-.2y}dxdy = .1\frac{1}{-.5}e^{-.5x}|_0^{\infty}\frac{1}{-.2}e^{-.2y}|_0^{\infty} = 1(0-1)(0-1) = 1$$

By definition $P(X \leq 2, Y \leq 5) = \int_0^2\int_0^5 .1e^{-.5x}e^{-.2y}\,dydx$ but this is

$$.1\frac{1}{-.5}e^{-.5x}|_0^2\frac{1}{-.2}e^{-.2y}|_0^5 = (e^{-1}-1)^2$$

Finally, $E(x) = \int_0^{\infty}\int_0^{\infty} f(x,y)x\,dxdy$. Since it's easy to integrate in y first, we do so, getting $E(x) = \frac{-.1}{.2}\int_0^{\infty} xe^{-.5x}\,dx$ by the formula given, this is $\frac{-1}{2}(\frac{1}{.5^2}(-.5x-1)e^{-.5x}|_0^{\infty} = \frac{-1}{2}\frac{-1}{.25} = 2$.