

# MTH 142 Practice Problems for Exam 3 -Spring 2004

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Sections 9.1, 9.2, 9.3, 9.4, 10.1, 10.2, 10.3, 10.4, 10.5

1. Obtain the first three nonzero terms of the Taylor series of  $f(x) = \sqrt{x}$  about 4.
2. Obtain  $P_3(x)$  = the Taylor polynomial of order 3 of  $\tan x$  about  $a = \pi/4$
3. Calculate the radius of convergence of the power series  $\sum_{n=0}^{\infty} n3^n(x-2)^n$ .
4. Given that  $R = 2$  is the radius of convergence of the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (x+1)^n$ , find the interval of convergence (include endpoint analysis).
5. Calculate the radius of convergence of the power series  $\sum_{n=0}^{\infty} \frac{n+1}{n+3} x^n$ .
6. Given that  $R = 1$  is the radius of convergence of the series  $\sum_{n=1}^{\infty} \frac{n}{n^2+1} x^n$ , find the interval of convergence (include endpoint analysis).
7. A certain amount of fresh water shrimp is placed in a tank together with 2 lbs. of food, at 12:00 noon on January 1st, and 2 lbs. of food are added to the tank at noontime every day after Jan 1. After every 24 hours, 15% of the food either decomposes or is eaten.
  - a) How much food is in the tank right after 12:00 noon on January 20 th? Give details, and explain how you arrived at your answer.
  - b) In the long run, how much food is in the tank right after noontime? (Section 9.1)
8. The degree 2 Taylor polynomial approximation of  $f(x) = \ln(1+x)$  for  $x$  near 0 is  $P_2(x) = x - \frac{1}{2}x^2$ . We wish to approximate  $\ln(1.5)$  by  $P_2(x)$ .
  - a) What should  $x$  be? What is the error? (use your calculator)
  - b) Find a bound for the error using the formula studied in class.
9.
  - a) Use the method of substitution to find the first 4 nonzero terms of the Taylor series of the function  $f(x) = \frac{1}{\sqrt{1+x^2}}$  about  $x = 0$ .
  - b) Use series to answer the following question: For values of  $x$  that are close to 0, which function is larger,  $\cos(x)$  or  $\frac{1}{\sqrt{1+x^2}}$ ?
10. Calculate the order 3 Fourier polynomial of the  $2\pi$  periodic function given on  $[-\pi, \pi)$  by

$$f(t) = \begin{cases} -0.5 & \text{if } -\pi \leq t < 0 \\ 1 & \text{if } 0 \leq t < \pi \end{cases}$$

11. Calculate the order 3 Fourier polynomial of the function given on  $[-\pi, \pi)$  by  $f(t) = 3t$ .

**Use the suggested method to determine if the series converges**

12.  $\sum_{n=2}^{\infty} \frac{n^2}{1+n^3}$  (a) Comparison Test. (b) Integral Test.

13.  $\sum_{n=1}^{\infty} 1 + \frac{1}{n}$  (a) comparison test. (b) integral test

14.  $\sum_{n=3}^{\infty} \frac{(-2)^{n+1}}{\pi^n}$  (a) Geometric series. (b) alternating series test. (c) ratio test.

## SOLUTION MTH142 Practice Problems for Exam 2

1.

$$T(x) = 2 + \frac{1}{4}(x - 4) - \frac{1}{64}(x - 4)^2 + \frac{1}{512}(x - 4)^3 + \dots$$

2.

$$P_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

3. Using the ratio test we have

$$L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{(n+1)3^{n+1}(x-2)^{n+1}}{n3^n(x-2)^n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)3|x-2|}{n} = 3|x-2|$$

Now  $L < 1$  when  $3|x-2| < 1$ , that is,  $|x-2| < \frac{1}{3}$ . Then the radius of convergence is  $\frac{1}{3}$ .

4. Given that  $R = 2$  is the radius of convergence, we note that the “base point” is  $a = -1$ , hence the interval of convergence goes from  $-3$  to  $1$ . We now analyze convergence at the endpoints.

(a) Substituting  $x = -3$  into  $\sum_{n=0}^{\infty} \frac{1}{n^2 2^n} (x+1)^n$ , we have the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

The absolute convergence test guarantees the convergence of the series provided the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, which does converge ( $p$ -series, with  $p = 2$ ). Hence  $x = -3$  belongs to the interval of convergence.

(b) Now substitute  $x = 1$  into  $\sum_{n=0}^{\infty} \frac{1}{n^2 2^n} (x+1)^n$ , to obtain the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} (2)^n = \sum_{n=1}^{\infty} \frac{1}{n^2}$  which converges ( $p = 2$  again). Conclusion: the interval of convergence is  $-3 \leq x \leq 1$  (or, in interval notation,  $[-3, 1]$ .)

5. Using the ratio test we have

$$L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{\frac{n+2}{n+4} x^{n+1}}{\frac{n+1}{n+3} x^n} \right| = \lim_{n \rightarrow \infty} \frac{(n+2)(n+3)|x|}{(n+1)(n+4)} = \lim_{n \rightarrow \infty} \frac{(n^2 + 5n + 6)|x|}{n^2 + 5n + 4} = |x|$$

Now  $L < 1$  when  $|x| < 1$ . Then the radius of convergence is  $1$ .

6. Given that  $R = 1$  is the radius of convergence, we note that the “base point” is  $a = 0$ , hence the interval of convergence goes from  $-1$  to  $1$ . We now analyze convergence at the endpoints.

Substituting  $x = -1$  into  $\sum_{n=0}^{\infty} \frac{n}{n^2 + 1} x^n$ , we obtain the series  $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1} (-1)^n$  which is an

alternating series. Since  $\frac{1}{1^2+1} \geq \frac{2}{2^2+1} \geq \frac{3}{3^2+1} \geq \dots$  and also  $\lim_{n \rightarrow \infty} \frac{n}{n^2+1} = 0$ , the series converges. Therefore  $x = -1$  is part of the interval of convergence.

To test the other endpoint, substitute  $x = 1$  into the power series to obtain  $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1} (1)^n = \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

This is a divergent series, since  $f(x) = \frac{x}{x^2+1}$  is decreasing and positive, we may apply the

integral test for series: we get:  $\int_1^{\infty} \frac{x}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[ \frac{1}{2} \ln |x^2 + 1| \right]_1^b =$

$\lim_{b \rightarrow \infty} \frac{1}{2} \ln |b^2 + 1| - \frac{1}{2} \ln |2| = \infty$  Therefore  $x = 1$  is not part of the interval of convergence.

We conclude that the interval of converge is  $-1 \leq x < 1$ , or in interval notation,  $[-1, 1)$ .