3.2 Predicates and Quantified Statements II

We define two quantified statements as Logically equivalent if they have identical truth values no matter what predicates are substituted for the predicate symbols, and no matter what sets are used for the domain.

Negations of Quantified Statements

Consider the statement "All students take calculus." What is the negation of this statement?

$\forall x \in \{\text{students}\}$, x takes calculus

negation: $\exists x \in \{\text{students}\} \mid$ x does not take calculus.

Definition

- The negation of a statement of the form

$$\forall x \in D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \mid \sim Q(x)$$

Consider the statement "Someone in this class is allergic to chocolate." What is the negation of this statement?

no one

$\exists x \in \underbrace{\{\text{students of this class}\}}_{D} \mid \underbrace{x \text{ is allergic to chocolate}}_{P(x)}$.

negation: $\forall x \in D, \sim P(x)$

Definition

- The negation of a statement of the form

$$\exists x \in D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim Q(x)$$

☆ Switch quantifier and negate predicate.

Example 1

Write negations for the following statements:

1. $\forall x \in \mathbb{R}, x^2 > 1$

   $\exists x \in \mathbb{R} \mid x^2 \leq 1$

2. $\exists y \in \mathbb{Z}$ such that $y^2 = -7$

   $\forall y \in \mathbb{Z}, y^2 \neq -7$

3. No mathematicians are interesting.

   $\forall x \in \{\text{mathematicians}\}$, x is not interesting

   negate: $\exists x \in \{\text{math.}\} \mid$ x is interesting

   Some math. are interesting.

4. Some athletes are over the age of 40.

   $\exists x \in \{\text{athletes}\} \mid$ x is over the age of 40

   negate: $\forall x \in \{\text{athletes}\}$, x is under the age of 40

   All athletes are under the age of 40.

5. The number 1357 is divisible by some integer between 1 and 37.

   All integers between 1 and 37 are not divisible by the number 1357.

Negating Condition Statements

Recall that

$$\sim (P \to Q) \equiv P(x) \wedge \sim Q(x).$$

Definition

The negation of a universal conditional statement is of the form

$$\begin{aligned} \sim (\forall x,\ \text{if } P(x),\ \text{then } Q(x)) &\equiv \exists x \text{ such that } \sim (P(x) \to Q(x)) \\ &\equiv \exists x \text{ such that } P(x) \wedge \sim Q(x) \end{aligned}$$

Example 2

Write negations for the following universal conditional statements.

1. $\forall x$, if $x < -1$, then $x^2 > 1$

   Neg: $\exists x \mid (x < -1) \wedge (x^2 \leq 1)$

   or

   $(x < -1) \wedge \sim (x^2 > 1)$

2. All actors making more than 26 million dollars will adopt a child from another country.

   $\forall x \in \{\text{actors}\}$, if x makes more than 26 mil dollars, then x will adopt a child from another country.

   Neg: $\exists x \in \{\text{actor}\} \mid$ x makes more than 26 million and x did not adopt a child from another country.

   Some actor make more than 26 million and will not adopt a child from another country.

Note: We can relate $\forall$ and $\wedge$ by noting that

$$\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge ... \wedge Q(x_n),$$

where $x_i \in D$.

Similarly, we can relate $\exists$ and $\vee$ by noting that

$$\exists x \in D \text{ such that } Q(x) \equiv Q(x_1) \vee Q(x_2) \vee ... \vee Q(x_n),$$

where $x_i \in D$.

Example 3

Rewrite the following in terms of $\wedge$ or $\vee$.

1. $\forall$ binary digits x, $x^3 = x$.

   $\equiv (0^3 = 0) \wedge (1^3 = 1)$

2. $\exists$ a binary digit x such that $x + x = x$.

   $\equiv (0 + 0 = 0) \vee (1 + 1 = 1)$

Note that we say that the statement $\forall x \in D$, if $P(x)$, then $Q(x)$ is vacuously true if, and only if, $P(x)$ is false for all x in our domain D. For example, suppose I said that all Jersey Shore cast members with a master's degree will go on to become astronauts. This is vacuously true, since none of the cast members have a master's degree.