Math 21b: Practice questions for second midterm

1. Let A be the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix},$$

and let $\vec{b}_1, \vec{b}_2$ be the vectors

$$\vec{b}_1 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

   (a) Compute $\det(A)$. What does this tell you about solutions of the linear system $A\vec{x} = \vec{b}$ for arbitrary $\vec{b}$?

   (b) Does $A\vec{x} = \vec{b}_1$ have an exact solution?

   (c) Does $A\vec{x} = \vec{b}_2$ have an exact solution?

   (d) Find an orthonormal basis for $\text{im}(A)$.

   (e) Find a least-squares solution of $A\vec{x} = \vec{b}_1$.

   (f) Find a least-squares solution of $A\vec{x} = \vec{b}_2$.

2. (a) [1 pt.] For a square matrix A, define the *characteristic polynomial* $f_A(\lambda)$, the *algebraic multiplicity* of an eigenvalue, and the *geometric multiplicity* of an eigenvalue.

   (b) [2 pts.] For any scalar a let A be the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & a & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Compute the characteristic polynomial $f_A(\lambda)$.

   (c) [4 pts.] For which value(s) of a does A have an eigenvalue λ of algebraic multiplicity at least 2? What is that λ in each case?

   (d) [2 pts.] For each of the cases you found in (c), what is the geometric multiplicity of λ?

   (e) [1 pt.] For each of the cases you found in (c), is A diagonalizable?

3. Let A be the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$.

   (a) Compute the eigenvalues of A.

   (b) Find an eigenbasis for A.

   (c) Find a matrix S such that $D = S^{-1}AS$ is diagonal. What is D?

   (d) Give a formula for A^n. Check your work by verifying that the cases $n = 1$ and $n = 2$ of your formula agree with A and A^2.

True or False?

- If $\vec{x}$ and $\vec{y}$ are any vectors in $\mathbb{R}^n$ then $\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$.
- If A is a matrix and $\vec{b}$ is a vector in $\ker A^T$ then $\vec{0}$ is a least-squares solution of $A\vec{x} = \vec{b}$.
- There exists a subspace V of $\mathbb{R}^3$ such that $\dim V^\perp = \dim V$.
- If a matrix A has $\operatorname{tr}(A) = 0$ then A is not invertible.
- The matrix $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 0 \\ 8 & 9 & 10 & 11 \end{bmatrix}$ has positive determinant.
- An $n \times n$ matrix can have at most n real eigenvalues.
- If A is a square matrix such that $A^m = 0$ for some m then 0 is an eigenvalue of A.
- If A is a square matrix such that $A^m = 0$ for some m then 0 is the only eigenvalue of A.