Applied Differential Equations 2280
Sample Final Exam
Wednesday, 6 May 2009, 7:30-10:15am

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Quadrature Equation)

Solve for the general solution $y(x)$ in the equation $y' = 2\cot x + \frac{1250x^3}{1+25x^2} + x\ln(1+x^2)$.

[The required integration talent includes basic formulae, integration by parts, substitution and college algebra.]

Answer:

$$y = 2\ln(\sin(x)) + \frac{49}{2}x^2 - \ln\left(1+25x^2\right) + 1/2\left(1+x^2\right)\ln\left(1+x^2\right) - 1/2 + C$$

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2. (Separable Equation Test)

The problem $y' = f(x,y)$ is said to be separable provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [75%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

☐	$y' = -y(2xy+1) + (2x+3)y^2$	☐	$yy' = xy^2 + 5x^2y$
☐	$y' = e^{x+y} + e^y$	☐	$3y' + 5y = 10y^2$

(b) [25%] State a test which can verify that an equation is not separable. Use the test to verify that $y' = x + \sqrt{|xy|}$ is not separable.

Answer:

(a) $yy' = xy^2 + 5x^2y$ is not separable, but the other three are separable.

(b) Test: f_y/f not independent of x implies not separable.

Let $f = x + \sqrt{|xy|}$ and assume $x > 0$. Then $y > 0$ and $f = x + \sqrt{xy}$. We have $f_y = 1/y$ and $f_y/f = y/(x+\sqrt{xy})$ depends on x, so the DE is not separable.

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3. (Solve a Separable Equation)

Given $y^2 y' = \dfrac{2x^2+3x}{1+x^2}\left(\dfrac{125}{64} - y^3\right)$.

(a) Find all equilibrium solutions.

(b) Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly.

Answer:

(a) $y = 5/4$

(b)

$$-\frac{1}{3}\ln|125 - 64y^3| = 2x + \frac{3}{2}\ln(1+x^2) - 2\arctan(x) + c$$

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4. (Linear Equations)

(a) [60%] Solve $2v'(t) = -32 + \dfrac{2}{3t+1}v(t)$, $v(0) = -8$. Show all integrating factor steps.

(b) [30%] Solve $2\sqrt{x+2}\,\dfrac{dy}{dx} = y$. The answer contains symbol c.

(c) [10%] The problem $2\sqrt{x+2}\,y' = y - 5$ can be solved using the answer y_h from part (b) plus superposition $y = y_h + y_p$. Find y_p. Hint: If you cannot write the answer in a few seconds, then return here after finishing all problems on the exam.

Answer:

(a) $v(t) = -24t - 8$

(b) $y(x) = Ce^{\sqrt{x+2}}$

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$dx/dt = 1000\left(2 - \sqrt[5]{x}\right)^3(2+3x)(9x^2-4)^8.$$

Expected in the diagram are equilibrium points and signs of x' (or flow direction markers $<$ and $>$).

(b) [40%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, node, source, sink, stable, unstable. Show at least 8 threaded curves. A direction field is not expected or required.

(c) [10%] Outline how to solve for non-equilibrium solutions, without doing any integrations or long details.

Answer:

(a) and (b) See a handwritten exam solution for a similar problem on midterm 1.
(c) Put the DE into the form $y'/G(y) = F(x)$ and then apply the method of quadrature.

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6. (ch3)
(a) Solve for the general solutions:

(a.1) [25%] $y'' + 4y' + 4y = 0$,

(a.2) [25%] $y^{vi} + 4y^{iv} = 0$,

(a.3) [25%] Char. eq. $r(r-3)(r^3 - 9r)^2(r^2+4)^3 = 0$.

(b) Given $6x''(t) + 7x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 6$, $c = 7$, $k = 2$, solve the differential equation [15%] and classify the answer as over-damped, critically damped or under-damped [5%]. Illustrate in a physical model drawing the meaning of constants m, c, k [5%].

Answer:
(a)
1: $r^2 + 4r + 4 = 0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
2: $r^{iv} + 4r^2 = 0$, roots $r = 0, 0, 2i, -2i$. Then $y = c_1e^{0x} + c_2xe^{0x} + c_3\cos 2x + c_4\sin 2x$.
3: Write as $r^3(r-3)^3(r+3)^2(r^2+4)^3 = 0$. Then y is a linear combination of the atoms 1, x, x^2, e^{3x}, xe^{3x}, x^2e^{3x}, e^{-3x}, xe^{-3x}, $\cos 2x$, $x\cos 2x$, $x^2\cos 2x$, $\sin 2x$, $x\sin 2x$, $x^2\sin 2x$.
Part (b)
Use $6r^2 + 7r + 2 = 0$ and the quadratic formula to obtain roots $r = -1/2, -2/3$. Then $x(t) = c_1e^{-t/2} + c_2e^{-2t/3}$. This is over-damped. The illustration shows a spring, dampener and mass with labels k, c, m, x and the equilibrium position of the mass.

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7. (ch3)
Determine for $y^{vi} + y^{iv} = x + 2x^2 + x^3 + e^{-x} + x\sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Answer:
The homogeneous solution is a linear combination of the atoms 1, x, x^2, x^3, $\cos x$, $\sin x$ because the characteristic polynomial has roots $0, 0, 0, 0, i, -i$.
$\boxed{1}$ An initial trial solution y is constructed for atoms 1, x, e^{3x}, e^{-3x}, $\cos x$, $\sin x$ giving

$$\begin{aligned}
y &= y_1 + y_2 + y_3 + y_4,\\
y_1 &= d_1 + d_2x + d_3x^2 + d_4x^3,\\
y_2 &= d_5\cos x + d_6x\cos x,\\
y_3 &= d_7\sin x + d_8x\sin x,\\
y_4 &= d_9e^{-x}.
\end{aligned}$$