1.6 Computing and Existence

The initial value problem

$$(1) \qquad y' = f(x, y), \quad y(x_0) = y_0$$

is studied here from a computational viewpoint. Answered are some basic questions about practical and theoretical computation of solutions:

- What in problem (1) causes numerical methods to fail?
- What hypotheses for (1) make numerical methods applicable?
- When does (1) have a symbolic solution?

Three Key Examples

The range of unusual behavior of solutions to $y' = f(x, y)$, $y(x_0) = y_0$ can be illustrated by three examples.

(A)	$y' = 3(y-1)^{2/3}$, $y(0) = 1$.	The right side $f(x, y)$ is continuous. It has two solutions $y = 1 + x^3$ and $y = 1$.
(B)	$y' = \dfrac{2y}{x-1}$, $y(0) = 1$.	The right side $f(x, y)$ is discontinuous. It has infinitely many piecewise-defined solutions $y = \begin{cases} (x-1)^2 & x < 1, \\ c(x-1)^2 & x \ge 1. \end{cases}$
(C)	$y' = 1 + y^2$, $y(0) = 0$.	The right side $f(x, y)$ is differentiable. It has unique solution $y = \tan(x)$, which satisfies $y(x_0) = \infty$ at time $x_0 = \pi/2$.

Numerical method failure can be caused by multiple solutions to problem (1), e.g., examples (A) and (B), because a numerical method is going to compute just *one* answer; see Example 32, page 63. Multiple solutions are often signaled by discontinuity of either f or its partial derivative f_y. In (A), the right side $3(y-1)^{2/3}$ has an infinite partial at $y = 1$, while in (B), the right side $2y/(x-1)$ is infinite at $x = 1$.

Simple jump discontinuities, or **switches**, appear in modern applications of differential equations. It is important therefore to allow $f(x, y)$ to be discontinuous, in a limited way, but multiple solutions must be avoided, e.g., example (B). An important success story in electrical engineering is circuit theory with periodic and piecewise-defined inputs. See Example 33, page 64.

Discontinuities of f or f_y in problem (1) should raise questions about the applicability of numerical methods. Exactly why there is not a precise and foolproof test to predict failure of a numerical method remains to be explained.

Theoretical solutions exist for problem (1), if $f(x, y)$ is *continuous*; see Peano's theorem, page 62. This solution may blow up in a finite interval, e.g., $y = \tan(x)$ in example (C); see Example 34, page 64.

No closed-form solution formula exists as a result of the basic theory. In part, this dilemma is due to the possibility of multiple solutions, if f is only continuous, e.g., example (A). Additional assumptions of a general nature do not seem to help: *there is in general no closed-form formula available for the solution.*

Exactly one theoretical solution exists in problem (1), provided $f(x, y)$ and $f_y(x, y)$ are continuous; see the Picard–Lindelöf theorem, page 62. The situation with numerical methods improves dramatically: the most popular methods are tractable.

Why Not "Put it on the computer?"

Typically, scientists and engineers rely upon computer algebra systems and numerical laboratories, e.g., `maple`, `mathematica` and `matlab`.

Computerization of differential equations constantly improves, with the advent of computer algebra systems and ever-improving numerical methods. Indeed, neither an advanced degree in mathematics nor a wizard's hat is required to query these systems for a closed-form solution formula. Many cases are checked systematically in a few seconds.

Fail-safe mechanisms usually do not exist for applying modern software to problem (1). The computer algebra system `maple` reports the "solution" $y = 1 + x^3$ for the problem $y' = 3(y - 1)^{2/3}$, $y(0) = 1$. But the obvious equilibrium solution $y = 1$ is unreported. The `maple` numeric solver silently accepts the same problem and solves for $y = 1$. To experience this, execute in `maple V Release 5.1` the code below.

```
de:=diff(y(x),x)=3*(y(x)-1)^(2/3):  ic:=y(0)=1:
dsolve({de,ic},y(x));                      # Symbolic sol
p:=dsolve({de,ic},y(x),numeric); p(1); # Numerical sol
```

There was an improvement in the report in later versions, e.g., version 10. In these versions, $y(x) = 1$ was reported for both. The inference for the maple user is that there is a unique solution, but the model has multiple solutions, making both reports incorrect.

Numerical instability is typically not reported by computer software. To understand the difficulty, consider the differential equation

$$y' = y - 2e^{-x}, \quad y(0) = 1.$$

The symbolic solution is $y = e^{-x}$. Attempts to solve the equation numerically will inevitably compute the nearby solutions $y = ce^x + e^{-x}$, where c is small. As x grows, the numerical solution grows like e^x, and $|y| \to \infty$. For example, `maple` computes $y(30) \approx -72557$, but $e^{-30} \approx 0.94 \times 10^{-13}$. In reality, the solution $y = e^{-x}$ cannot be computed. The `maple` code:

```
de:=diff(y(x),x)=y(x)-2*exp(-x):  ic:=y(0)=1:
p:=dsolve({de,ic},y(x),numeric):  p(30);
```

Mathematical model formulation, prior to using a computer, seems to be an essential skill which does not come in the colorfully decorated package from the software vendor. It is this creative skill that separates the practicing scientist from the person on the street who has enough money to buy a computer program.

Closed-Form Existence-Uniqueness Theory

The **closed-form existence-uniqueness theory** describes models

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2}$$

for which a closed-form solution is known, as an equation of some sort. The objective of the theory for first order differential equations is to obtain existence and uniqueness by exhibiting a solution formula. The mathematical literature which documents these models is too vast to catalog in a textbook. We discuss only the most popular models.

`Dsolve` Engine in `Maple`. This computer algebra system has an implementation for some specialized equations within the closed-form theory. Below are some of the equation types examined by `maple` for solving a differential equation using classification methods. Not everything tried by `maple` is listed, e.g., Lie symmetry methods, which are beyond the scope of this text.

Equation Type	**Differential Equation**
Quadrature	$y' = F(x)$
Linear	$y' + p(x)y = r(x)$
Separable	$y' = f(x)g(y)$
Abel	$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$
Bernoulli	$y' + p(x)y = r(x)y^n$
Clairaut	$y = xy' + g(y')$
d'Alembert	$y = xf(y') + g(y')$
Chini	$y' = f(x)y^n - g(x)y + h(x)$