

## Announcements

- Problem set 6 is due this Friday.

## Today's Agenda

- Additional Notes on Partial-Fraction Expansion
- Interpretation of Phase in the Frequency Response
  - Linear phase and constant time shift
  - Nonlinear phase and group delay
- First- and Second-Order Systems
- Bode Plots
  - First-order system Bode plots
  - Second-order system Bode plots

# 1 Additional Notes on Partial-Fraction Expansion

Last week, we saw that to do partial-fraction expansion on rational functions with repeated roots in the denominator, there was a complicated formula we could apply. However, this is confusing and non-intuitive. Fortunately, there is another method that is easy to follow.

Let's say we want to do a partial-fraction expansion on the following function with repeated roots in the denominator:

$$F(x) = \frac{2x + 1}{(x + 3)^3(x + 2)}.$$

The partial-fraction expansion will be in the form

$$\frac{2x + 1}{(x + 3)^3(x + 2)} = \frac{A_0}{(x + 3)^3} + \frac{A_1}{(x + 3)^2} + \frac{A_2}{(x + 3)} + \frac{B}{(x + 2)}.$$

We can determine that  $A_0 = 5$  and  $B = -3$  using last week's method (i.e., the cover up method). However, determining  $A_1$  and  $A_2$  using cover up results in the left side becoming  $\frac{2x+1}{0} = \infty$ . Fortunately, we can easily fix this by subtracting the highest-order term from both sides:

$$\frac{2x + 1}{(x + 3)^3(x + 2)} - \frac{5}{(x + 3)^3} = \frac{A_1}{(x + 3)^2} + \frac{A_2}{(x + 3)} + \frac{-3}{(x + 2)}.$$

Simplifying the left side, we see that the order of the denominator has been reduced:

$$\frac{-3}{(x + 3)^2(x + 2)} = \frac{A_1}{(x + 3)^2} + \frac{A_2}{(x + 3)} + \frac{-3}{(x + 2)}.$$

We can now determine that  $A_1 = 3$  using cover up as before. How do we determine  $A_2$ ? You guessed it. Just as before, we can subtract the highest-order term from both sides to reduce the order of the denominator. Using cover up, we can now determine that  $A_2 = 3$ . Therefore, the partial-fraction expansion of  $F(x)$  is

$$\frac{2x + 1}{(x + 3)^3(x + 2)} = \frac{5}{(x + 3)^3} + \frac{3}{(x + 3)^2} + \frac{3}{(x + 3)} + \frac{-3}{(x + 2)}.$$

So in general, to do partial-fraction expansion on rational functions with repeated roots in the denominator, determine all the highest-order residuals ( $A_0$ ,  $B_0$ ,  $C_0$ , etc.) using the cover up method, subtract the highest-order terms from both sides, simplify, and then recurse.

## 2 Interpretation of Phase in the Frequency Response

In past weeks we've seen that examining the *magnitude* of a system's frequency response  $|H(j\omega)|$  allows us to intuit the behavior of the system by matching it to one of several filter types. (For review, those filter types were lowpass, highpass, bandpass, bandstop, and all-pass.) We'll now see that we can also interpret the *phase* of a system's frequency response  $\angle H(j\omega)$ .

### 2.1 Linear phase and constant time shift

We know from the Fourier transform representation of signals that certain CT signals can be expressed as superpositions of complex exponentials of the form  $e^{j\omega t}$  (without loss of generality, we'll use CT notation here). Suppose such a signal  $x(t)$  has Fourier transform  $X(j\omega)$ . We know from the transform tables that the FT of the shifted version of the signal  $x(t - t_0)$  is  $e^{-j\omega t_0} X(j\omega)$ . But if we lost our tables and even forgot the formula for the Fourier transform, how can we re-derive this? And what does it mean?

Each of the complex exponentials  $e^{j\omega t}$  has a cosine in the real component and a sine in the imaginary component. If we time delay the entire signal  $x(t)$  by  $t_0$ , then each exponential would also be delayed by the same amount in *time*. However, they would *not* be shifted by the same amount in *phase*. Let's consider the cosine component  $x_c(t) = \cos(\omega t)$ . A time shift produces  $x_c(t - t_0) = \cos(\omega(t - t_0)) = \cos(\omega t - \theta)$ , where  $\theta = \omega t_0$  is the *phase shift* for that signal. Note that the phase added is proportional to the frequency of the cosine. This is consistent with our idea that a constant time shift corresponds to a larger phase shift for high-frequency components than it does for low-frequency ones (recall the diagrams drawn in recitation and tutorial). So, the complex exponential  $e^{j\omega t}$  gets mapped to  $e^{j\omega(t-t_0)} = e^{j\omega t} e^{-j\omega t_0}$ . Since the Fourier transform  $X(j\omega)$  of a signal is simply the scale factor (ignoring the  $2\pi$  factor...) that sits in front of the complex exponentials when they are integrated together to form  $x(t)$ , this means that a time shift multiplies the FT by  $e^{-j\omega t_0}$ . But we know that the phase of a complex number is the  $\theta$  in the  $e^{j\theta}$  portion of the number in the magnitude-phase representation  $z = r e^{j\theta}$ . Multiplying complex numbers adds their phases, so we have:

A constant shift in time corresponds to the addition of a linear phase in frequency.