AAE 340 Problem Set 10 Solution – by Matt Gerberich

Problem 1)

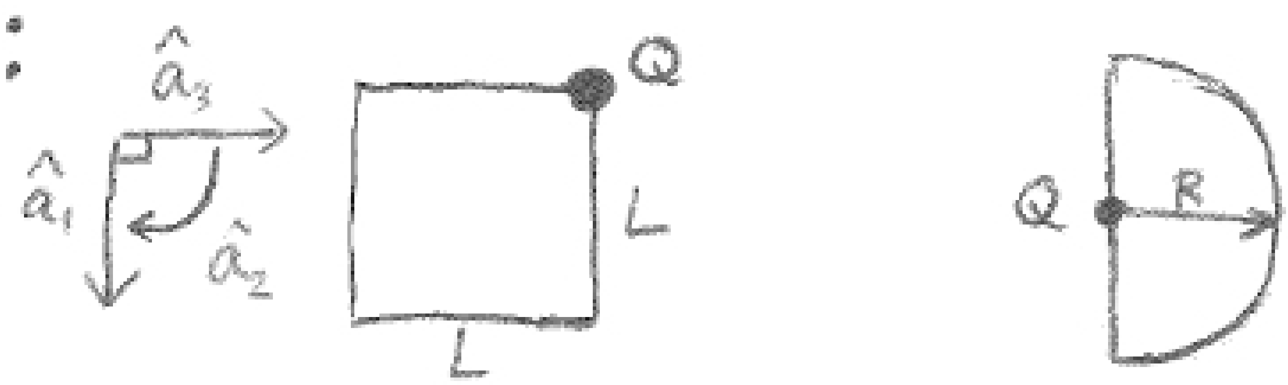
Given: — Square thin plate (total mass m, uniform density, side length L)
— Semicircular thin plate (total mass m, uniform density, radius R)
— Unit vectors $\hat{a}_i$ fixed in each plate
— Inertia table on Blackboard

Find: (i) $[I^Q_{\hat{a}}]$ for the square given point Q

$[I^Q_{\hat{a}}]$ for the semicircle given point Q

(ii) Check answers with inertia table

Diagram:



Analysis:

(i)

$$[I^Q_{\hat{a}}] = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

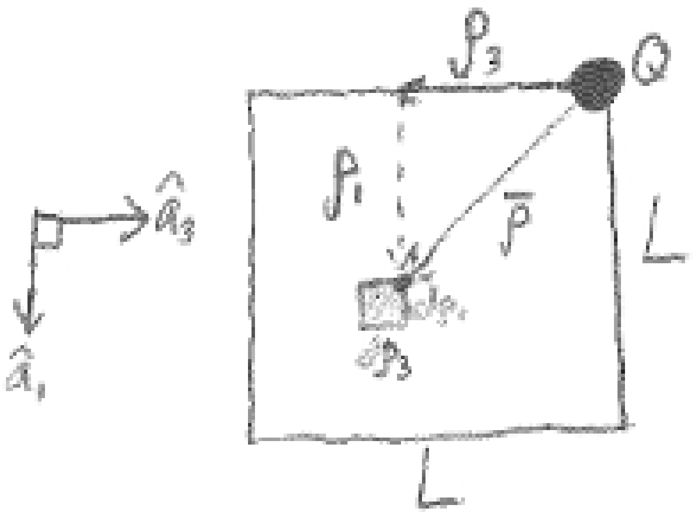
where

$$I_{11} = \int (\rho_2^2 + \rho_3^2)\, dm \qquad I_{12} = I_{21} = -\int \rho_1 \rho_2 \, dm$$

$$I_{22} = \int (\rho_1^2 + \rho_3^2)\, dm \qquad I_{13} = I_{31} = -\int \rho_1 \rho_3 \, dm$$

$$I_{33} = \int (\rho_1^2 + \rho_2^2)\, dm \qquad I_{23} = I_{32} = -\int \rho_2 \rho_3 \, dm$$

(a) For the square –



$\bar{\rho} = \rho_1 \hat{a}_1 + \rho_3 \hat{a}_3$

Since this is a thin plate, we will proceed with $\rho_2 = 0$

Let the uniform density be $\nu = \frac{m}{L^2}$

Also, $dm = \nu \, d\rho_1 \, d\rho_3$

$$I_{11} = \int (\rho_2^2 + \rho_3^2)\, dm = \int_{-L}^{0}\int_{0}^{L} (0^2 + \rho_3^2)\, \nu \, d\rho_1 \, d\rho_3 = \nu \int_{-L}^{0}\int_{0}^{L} \rho_3^2 \, d\rho_1 \, d\rho_3$$

$$= \nu \int_{-L}^{0} \left[\rho_3^2 \rho_1\right]_{\rho_1=0}^{L} d\rho_3 = \nu \int_{-L}^{0} \rho_3^2 L \, d\rho_3 = \nu L \left[\tfrac{1}{3}\rho_3^3\right]_{-L}^{0} = \nu L \cdot \tfrac{1}{3} L^3$$

Since $\nu = \frac{m}{L^2}$, $I_{11} = \frac{1}{3}\left(\frac{m}{L^2}\right) L^4 \longrightarrow I_{11} = \frac{1}{3} m L^2$

$$I_{22} = \int (\rho_1^2 + \rho_3^2)\, dm = \int_{-L}^{0}\int_{0}^{L} (\rho_1^2 + \rho_3^2)\, \nu \, d\rho_1 \, d\rho_3 = \nu \int_{-L}^{0} \left[\tfrac{1}{3}\rho_1^3 + \rho_3^2 \rho_1\right]_{0}^{L} d\rho_3$$

$$= \nu \int_{-L}^{0} \left[\tfrac{1}{3}L^3 + L\rho_3^2\right] d\rho_3 = \nu \left[\tfrac{1}{3}L^3 \rho_3 + \tfrac{1}{3} L \rho_3^3\right]_{-L}^{0} = \nu \left[\tfrac{1}{3}L^4 + \tfrac{1}{3}L^4\right] = \tfrac{2}{3}\nu L^4$$

Since $\nu = \frac{m}{L^2}$, $I_{22} = \frac{2}{3}\left(\frac{m}{L^2}\right) L^4 \longrightarrow I_{22} = \frac{2}{3} m L^2$

$$I_{33} = \int (\rho_1^2 + \rho_2^2)\, dm = \int_{-L}^{0}\int_{0}^{L} (\rho_1^2 + 0^2)\, \nu \, d\rho_1 \, d\rho_3 = \nu \int_{-L}^{0}\int_{0}^{L} \rho_1^2 \, d\rho_1 \, d\rho_3$$

$$= \nu \int_{-L}^{0} \left[\tfrac{1}{3}\rho_1^3\right]_{0}^{L} d\rho_3 = \nu \int_{-L}^{0} \tfrac{1}{3} L^3 \, d\rho_3 = \tfrac{1}{3}\nu L^3 \left[\rho_3\right]_{-L}^{0} = \tfrac{1}{3}\nu L^3 (L)$$

Since $\nu = \frac{m}{L^2}$, $I_{33} = \frac{1}{3}\left(\frac{m}{L^2}\right) L^4 \longrightarrow I_{33} = \frac{1}{3} m L^2$

$$I_{21} = I_{12} = -\int \rho_1 \rho_2 \, dm = -\int_{-L}^{0}\int_{0}^{L} \rho_1 \cdot 0 \, \nu \, d\rho_1 \, d\rho_3 = -\nu \int_{-L}^{0}\int_{0}^{L} 0 \, d\rho_1 \, d\rho_3 = 0$$

$$I_{31} = I_{13} = -\int \rho_1 \rho_3 \, dm = -\int_{-L}^{0}\int_{0}^{L} \rho_1 \rho_3 \, \nu \, d\rho_1 \, d\rho_3 = -\nu \int_{-L}^{0} \left[\tfrac{1}{2}\rho_1^2 \rho_3\right]_{0}^{L} d\rho_3$$

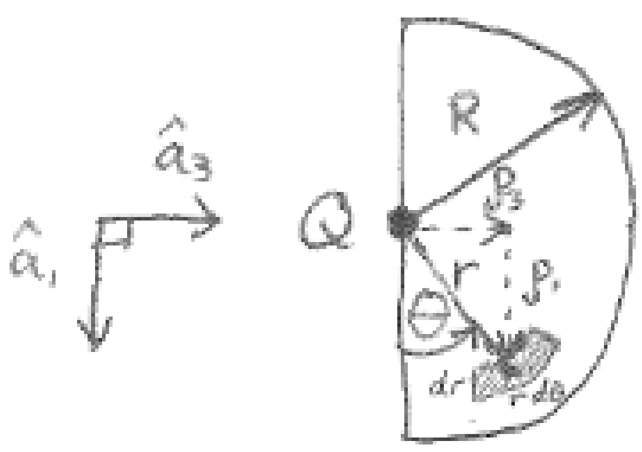
$$= -\nu \int_{-L}^{0} \left[\tfrac{1}{2}L^2 \rho_3\right] d\rho_3 = -\tfrac{1}{2}\nu L^2 \left[\tfrac{1}{2}\rho_3^2\right]_{-L}^{0} = -\tfrac{1}{2}\nu L^2 \left[0 - \tfrac{1}{2}L^2\right] = \tfrac{1}{4}\nu L^4$$

Since $\nu = \frac{m}{L^2}$, $I_{13} = \frac{1}{4}\left(\frac{m}{L^2}\right) L^4 \longrightarrow I_{13} = \frac{1}{4} m L^2$

$$I_{32} = I_{23} = -\int \rho_2 \rho_3 \, dm = -\int_{-L}^{0}\int_{0}^{L} 0 \cdot \rho_3 \, \nu \, d\rho_1 \, d\rho_3 = -\nu \int_{-L}^{0}\int_{0}^{L} 0 \, d\rho_1 \, d\rho_3 = 0$$

For the square, $$\boxed{[I^Q]_{\hat{a}} = \begin{bmatrix} \frac{1}{3}mL^2 & 0 & \frac{1}{4}mL^2 \\ 0 & \frac{2}{3}mL^2 & 0 \\ \frac{1}{4}mL^2 & 0 & \frac{1}{3}mL^2 \end{bmatrix}}$$

(b) For the semicircle —



As before, $\bar{\rho} = \rho_1 \hat{a}_1 + \rho_3 \hat{a}_3$, $\rho_2 = 0$.

The uniform density is again mass per area

However, the area of the semicircle is $\frac{1}{2}\pi R^2$

So, $\nu = \frac{m}{\frac{1}{2}\pi R^2} = \frac{2m}{\pi R^2}$

We can choose to calculate the moment of inertia matrix using polar coordinates

This means the infinitesimal mass is $dm = \nu\, dA = \nu\, r\, dr\, d\theta$

With θ defined as zero in the $\hat{a}_3$ direction, and positive in the counter-clockwise direction,

we can rewrite $\rho_1 = r\cos\theta$, $\rho_3 = r\sin\theta$

and our limits are $0 \le r \le R$, $0 \le \theta \le \pi$

$$I_{11} = \int(\rho_2^2 + \rho_3^2)\,dm = \int_0^\pi\int_0^R [0 + (r\sin\theta)^2]\,\nu\, r\, dr\, d\theta = \nu\int_0^\pi\int_0^R r^3 \sin^2\theta\, dr\, d\theta$$

$$= \nu\int_0^\pi \left[\frac{1}{4}r^4\sin^2\theta\right]_0^R d\theta = \frac{1}{4}R^4\nu\int_0^\pi \sin^2\theta\, d\theta = \frac{1}{4}R^4\nu\left[\frac{1}{2}(\theta - \sin\theta\cos\theta)\right]_0^\pi$$

$$= \frac{1}{4}R^4\nu\left[\frac{1}{2}(\pi - (0)(-1)) - \frac{1}{2}(0 - (0)(1))\right] = \frac{1}{8}R^4\nu\pi$$

Plugging in ν gives $I_{11} = \frac{1}{8}R^4\left(\frac{2m}{\pi R^2}\right)\pi = \frac{1}{4}mR^2$

$$I_{22} = \int(\rho_1^2 + \rho_3^2)\,dm = \int_0^\pi\int_0^R [(r\cos\theta)^2 + (r\sin\theta)^2]\,\nu\, r\, dr\, d\theta = \nu\int_0^\pi\int_0^R r^3\, dr\, d\theta$$

$$= \nu\int_0^\pi \left[\frac{1}{4}r^4\right]_0^R d\theta = \frac{1}{4}\nu R^4\int_0^\pi d\theta = \frac{1}{4}R^4\nu\pi$$

Plugging in ν gives $I_{22} = \frac{1}{4}R^4\left(\frac{2m}{\pi R^2}\right)\pi = \frac{1}{2}mR^2$