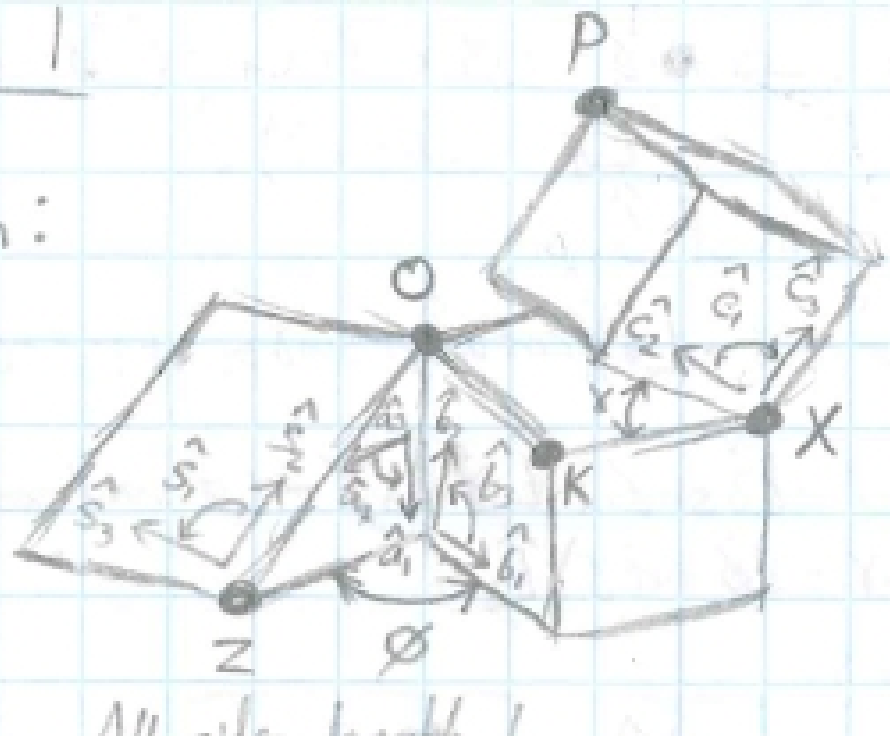
AAE 340 - PS2 F13 Matt Gerberich

Problem 1

Given:



All sides length L.

Find: (a) ${}^a\bar{V}^{KP}$ in terms of $\hat{b}$; generic?
${}^a\bar{A}^{KP}$ in terms of $\hat{b}$ and $\hat{c}$

(b) ${}^s\bar{V}^{K}$ and ${}^c\bar{V}^{K}$ in terms of $\hat{c}$. ${}^s\bar{V}^{K} = {}^c\bar{V}^{K}$?

(c) ${}^a\bar{A}^{P}$, $\frac{{}^b d^2 \bar{r}^{KP}}{dt^2}$, $\frac{{}^a d^2 \bar{r}^{KP}}{dt^2}$. Are these equivalent?

Solution:

(a) The velocity ${}^a\bar{V}^{KP} = \frac{{}^a d}{dr}\left(\bar{r}^{KP}\right)$

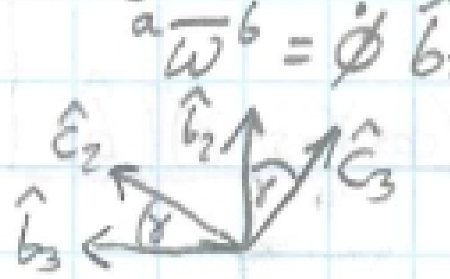
K and P are in two reference frames, b and c respectively.
If we define a new point X, as shown in the graph, we can write

$$\bar{r}^{KP} = \bar{r}^{KX} + \bar{r}^{XP} = \left(L(-\hat{b}_3)\right) + \left(L\hat{c}_2 + L\hat{c}_1 + L\hat{c}_3\right)$$

Seperating terms: $\frac{{}^a d}{dr}\left(\bar{r}^{KP}\right) = \frac{{}^a d}{dr}\left(\bar{r}^{KX}\right) + \frac{{}^a d}{dr}\left(\bar{r}^{XP}\right)$. Then the BKE gives

$$= \left[\frac{{}^b d}{dr}\left(\bar{r}^{KX}\right) + {}^a\bar{\omega}^b \times \bar{r}^{KX}\right] + \left[\frac{{}^c d}{dr}\left(\bar{r}^{XP}\right) + {}^a\bar{\omega}^c \times \bar{r}^{XP}\right]$$

${}^a\bar{\omega}^b = \dot{\phi}\,\hat{b}_2$; By the chain rule, ${}^a\bar{\omega}^c = {}^a\bar{\omega}^b + {}^b\bar{\omega}^c = \dot{\phi}\,\hat{b}_2 + \dot{\gamma}\,\hat{c}_1$



We can write the coordinate transformations between $\hat{b}$ and $\hat{c}$

$$\hat{b}_1 = -\hat{c}_1 \qquad \hat{c}_1 = -\hat{b}_1$$
$$\hat{b}_2 = \cos\gamma\,\hat{c}_3 + \sin\gamma\,\hat{c}_2 \qquad \hat{c}_2 = \cos\gamma\,\hat{b}_3 + \sin\gamma\,\hat{b}_2$$
$$\hat{b}_3 = \cos\gamma\,\hat{c}_2 - \sin\gamma\,\hat{c}_3 \qquad \hat{c}_3 = \cos\gamma\,\hat{b}_2 - \sin\gamma\,\hat{b}_3$$

Then, $\frac{{}^b d}{dr}\left(\bar{r}^{KX}\right) = \frac{{}^b d}{dr}\left(-L\hat{b}_3\right) = 0$, $\frac{{}^c d}{dr}\left(\bar{r}^{XP}\right) = \frac{{}^c d}{dr}\left(L\hat{c}_2 + L\hat{c}_1 + L\hat{c}_3\right) = 0$

$${}^a\bar{\omega}^b \times \bar{r}^{KX} = (\dot{\phi}\,\hat{b}_2) \times (-L\hat{b}_3) = -L\dot{\phi}\,\hat{b}_1$$

$${}^a\bar{\omega}^c \times \bar{r}^{XP} = \left[\dot{\phi}\,(\cos\gamma\,\hat{c}_3 + \sin\gamma\,\hat{c}_2) + \dot{\gamma}\,\hat{c}_1\right] \times \left[L\hat{c}_2 + L\hat{c}_1 + L\hat{c}_3\right]$$

(1)(a)

$$^a\bar{\omega}^c \times \bar{r}^{XP} = \begin{vmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_3 \\ \dot{\gamma} & \dot{\phi}\sin\gamma & \dot{\phi}\cos\gamma \\ L & L & L \end{vmatrix} = (L\dot{\phi}\sin\gamma - L\dot{\phi}\cos\gamma)\,\hat{c}_1 - (L\dot{\gamma} - L\dot{\phi}\cos\gamma)\,\hat{c}_2 + (L\dot{\gamma} - L\dot{\phi}\sin\gamma)\,\hat{c}_3$$

However, we wish to express this in terms of $\hat{b}$, so

$$^a\bar{\omega}^c \times \bar{r}^{XP} = (L\dot{\phi}\sin\gamma - L\dot{\phi}\cos\gamma)(-\hat{b}_1) + (L\dot{\phi}\cos\gamma - L\dot{\gamma})(\cos\gamma\,\hat{b}_3 + \sin\gamma\,\hat{b}_2) + (L\dot{\gamma} - L\dot{\phi}\sin\gamma)(\cos\gamma\,\hat{b}_2 - \sin\gamma\,\hat{b}_3)$$

$$= (L\dot{\phi}\cos\gamma - L\dot{\phi}\sin\gamma)\,\hat{b}_1 + (L\dot{\phi}\cos\gamma\sin\gamma - L\dot{\gamma}\sin\gamma + L\dot{\gamma}\cos\gamma - L\dot{\phi}\sin\gamma\cos\gamma)\,\hat{b}_2 + (L\dot{\phi}\cos^2\gamma - L\dot{\gamma}\cos\gamma - L\dot{\gamma}\sin\gamma + L\dot{\phi}\sin^2\gamma)\,\hat{b}_3$$

Adding terms together and simplifying gives:

$$\boxed{^a\bar{V}^{KP} = L\dot{\phi}(\cos\gamma - \sin\gamma - 1)\,\hat{b}_1 + L\dot{\gamma}(\cos\gamma - \sin\gamma)\,\hat{b}_2 + L(\dot{\phi} - \dot{\gamma}(\cos\gamma + \sin\gamma))\,\hat{b}_3}$$

This is not generic because the base point is not fixed in the reference frame a.

$$^a\bar{A}^{KP} = \frac{^a d}{dt}\left(^a\bar{V}^{KP}\right) = \frac{^b d}{dt}\left(^a\bar{V}^{KP}\right) + {^a\bar{\omega}^b} \times {^a\bar{V}^{KP}}$$

$$\frac{^b d}{dt}\left(^a\bar{V}^{KP}\right) = \left[L\ddot{\phi}(\cos\gamma - \sin\gamma - 1) + L\dot{\phi}\dot{\gamma}(-\sin\gamma - \cos\gamma)\right]\hat{b}_1 + \left[L\ddot{\gamma}(\cos\gamma - \sin\gamma) + L\dot{\gamma}^2(-\sin\gamma - \cos\gamma)\right]\hat{b}_2 + \left[L\ddot{\phi} - L\ddot{\gamma}(\cos\gamma + \sin\gamma) - L\dot{\gamma}^2(\cos\gamma - \sin\gamma)\right]\hat{b}_3$$

$$^a\bar{\omega}^b \times {^a\bar{V}^{KP}} = \left[\dot{\phi}\,\hat{b}_2\right] \times \left[L\dot{\phi}(\cos\gamma - \sin\gamma - 1)\,\hat{b}_1 + L\dot{\gamma}(\cos\gamma - \sin\gamma)\,\hat{b}_2 + L(\dot{\phi} - \dot{\gamma}(\cos\gamma + \sin\gamma))\,\hat{b}_3\right]$$

$\hat{b}_2 \times \hat{b}_1 = -\hat{b}_3$, $\hat{b}_2 \times \hat{b}_2 = 0$, $\hat{b}_2 \times \hat{b}_3 = \hat{b}_1$

$$= L\dot{\phi}^2(1 + \sin\gamma - \cos\gamma)\,\hat{b}_3 + 0 + L(\dot{\phi}^2 - \dot{\gamma}\dot{\phi}(\cos\gamma + \sin\gamma))\,\hat{b}_1$$

Combining terms:

In $\hat{b}$:

$$\boxed{\begin{aligned} ^a\bar{A}^{KP} &= L\left[\ddot{\phi}(\cos\gamma - \sin\gamma - 1) + \dot{\phi}^2 - 2\dot{\gamma}\dot{\phi}(\cos\gamma + \sin\gamma)\right]\hat{b}_1 \\ &+ L\left[\ddot{\gamma}(\cos\gamma - \sin\gamma) - \dot{\gamma}^2(\sin\gamma + \cos\gamma)\right]\hat{b}_2 \\ &+ L\left[\ddot{\phi} + \dot{\phi}^2(1 + \sin\gamma - \cos\gamma) - \ddot{\gamma}(\cos\gamma + \sin\gamma) - \dot{\gamma}^2(\cos\gamma - \sin\gamma)\right]\hat{b}_3 \end{aligned}}$$

If we go back and note that $^a\bar{\omega}^b \times \bar{r}^{KX} = -L\dot{\phi}\,\hat{b}_1 = L\dot{\phi}\,\hat{c}_1$, then we can write

$$^a\bar{V}^{KP} = L\dot{\phi}(1 + \sin\gamma - \cos\gamma)\,\hat{c}_1 + L(\dot{\phi}\cos\gamma - \dot{\gamma})\,\hat{c}_2 + L(\dot{\gamma} - \dot{\phi}\sin\gamma)\,\hat{c}_3$$

The BKE gives: $^a\bar{A}^{KP} = \frac{^a d}{dt}\left(^a\bar{V}^{KP}\right) = \frac{^c d}{dt}\left(^a\bar{V}^{KP}\right) + {^a\bar{\omega}^c} \times {^a\bar{V}^{KP}}$

$$\frac{^c d}{dt}\left(^a\bar{V}^{KP}\right) = \left[L\ddot{\phi}(1 + \sin\gamma - \cos\gamma) + L\dot{\phi}\dot{\gamma}(\cos\gamma + \sin\gamma)\right]\hat{c}_1 + \left[L\ddot{\phi}\cos\gamma - L\dot{\phi}\dot{\gamma}\sin\gamma - L\ddot{\gamma}\right]\hat{c}_2 + \left[L\ddot{\gamma} - L\ddot{\phi}\sin\gamma - L\dot{\phi}\dot{\gamma}\cos\gamma\right]\hat{c}_3$$

(1) (a) $^a\bar{\omega}^c \times {}^a\bar{V}^{KP} = \left[\dot{\gamma}\hat{c}_1 + \dot{\phi}\sin\gamma\,\hat{c}_2 + \dot{\phi}\cos\gamma\,\hat{c}_3\right] \times \left[L\dot{\phi}(1+\sin\gamma-\cos\gamma)\hat{c}_1\right.$

$$\left. + L(\dot{\phi}\cos\gamma - \dot{\gamma})\hat{c}_2 + L(\dot{\gamma} - \dot{\phi}\sin\gamma)\hat{c}_3\right]$$

$$= \begin{vmatrix} \hat{c}_1 & \hat{c}_2 & \hat{c}_3 \\ \dot{\gamma} & \dot{\phi}\sin\gamma & \dot{\phi}\cos\gamma \\ L\dot{\phi}(1+\sin\gamma-\cos\gamma) & L(\dot{\phi}\cos\gamma-\dot{\gamma}) & L(\dot{\gamma}-\dot{\phi}\sin\gamma) \end{vmatrix}$$

$$= L(\dot{\phi}\dot{\gamma}\sin\gamma - \dot{\phi}^2\sin^2\gamma - \dot{\phi}^2\cos^2\gamma + \dot{\phi}\dot{\gamma}\cos\gamma)\hat{c}_1 - L(\dot{\gamma}^2 - \dot{\phi}\dot{\gamma}\sin\gamma - \dot{\phi}^2\cos\gamma(1+\sin\gamma-\cos\gamma))\hat{c}_2$$

$$+ L(\dot{\phi}\dot{\gamma}\cos\gamma - \dot{\gamma}^2 - \dot{\phi}^2\sin\gamma(1+\sin\gamma-\cos\gamma))\hat{c}_3$$

Combining terms gives:

In $\hat{c}$:

$$\boxed{\begin{aligned} {}^a\bar{A}^{KP} &= L\left[\ddot{\phi}(1+\sin\gamma-\cos\gamma) - \dot{\phi}^2 + 2\dot{\phi}\dot{\gamma}(\sin\gamma+\cos\gamma)\right]\hat{c}_1 \\ &+ L\left[\ddot{\phi}\cos\gamma - \ddot{\gamma} - \dot{\gamma}^2 - \dot{\phi}^2\cos\gamma(1+\sin\gamma-\cos\gamma)\right]\hat{c}_2 \\ &+ L\left[\ddot{\gamma} - \ddot{\phi}\sin\gamma - \dot{\gamma}^2 - \dot{\phi}^2\sin\gamma(1+\sin\gamma-\cos\gamma)\right]\hat{c}_3 \end{aligned}}$$

(b) The lack of a base point for $^s\bar{V}^K$ and $^c\bar{V}^K$ implies they are generic, and have base points fixed in the s-frame and c-frame, respectively.

We will pick point O in the s-frame and point X in the c-frame as base points, such that $^s\bar{V}^K = \frac{^s d}{dt}(\bar{r}^{OK})$ and $^c\bar{V}^K = \frac{^c d}{dt}(\bar{r}^{XK})$

$\bar{r}^{OK} = L\hat{b}_1$, $\bar{r}^{XK} = L\hat{b}_3$

Using the BKE, $^s\bar{V}^K = \frac{^b d}{dt}(\bar{r}^{OK}) + {}^s\bar{\omega}^b \times \bar{r}^{OK}$

$^s\bar{\omega}^b = \dot{\phi}\hat{b}_2$, $\frac{^b d}{dt}(L\hat{b}_1) = 0$, $\dot{\phi}\hat{b}_2 \times L\hat{b}_1 = -L\dot{\phi}\hat{b}_3$

Since $\hat{b}_3 = \cos\gamma\,\hat{c}_2 - \sin\gamma\,\hat{c}_3$, $\boxed{^s\bar{V}^K = L\dot{\phi}\sin\gamma\,\hat{c}_3 - L\dot{\phi}\cos\gamma\,\hat{c}_2}$

Using the BKE, $^c\bar{V}^K = \frac{^c d}{dt}(\bar{r}^{XK}) + {}^c\bar{\omega}^b \times \bar{r}^{XK}$

$^c\bar{\omega}^b = \dot{\gamma}\hat{b}_1$, $\frac{^c d}{dt}(L\hat{b}_3) = 0$, $(\dot{\gamma}\hat{b}_1)\times(L\hat{b}_3) = -L\dot{\gamma}\hat{b}_2$

Since $\hat{b}_2 = \cos\gamma\,\hat{c}_3 + \sin\gamma\,\hat{c}_2$, $\boxed{^c\bar{V}^K = -L\dot{\gamma}\cos\gamma\,\hat{c}_3 - L\dot{\gamma}\sin\gamma\,\hat{c}_2}$

Although these are both generic velocities of K, they are not the same, because they are taken with respect to different reference frames.