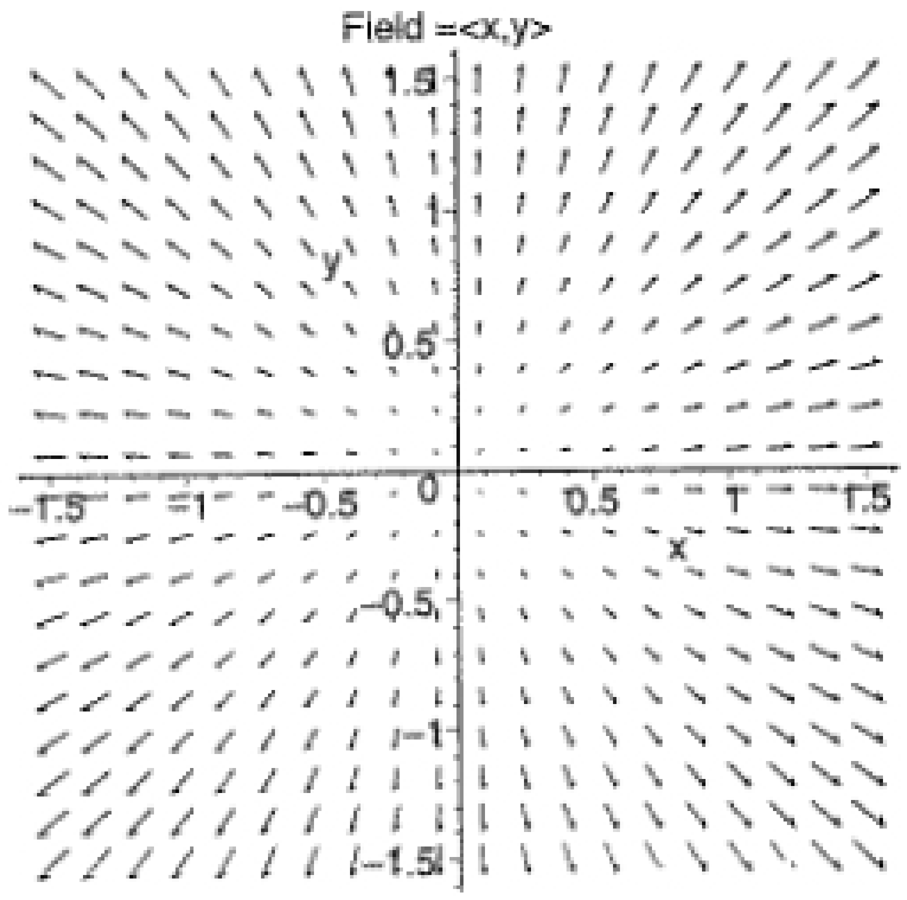
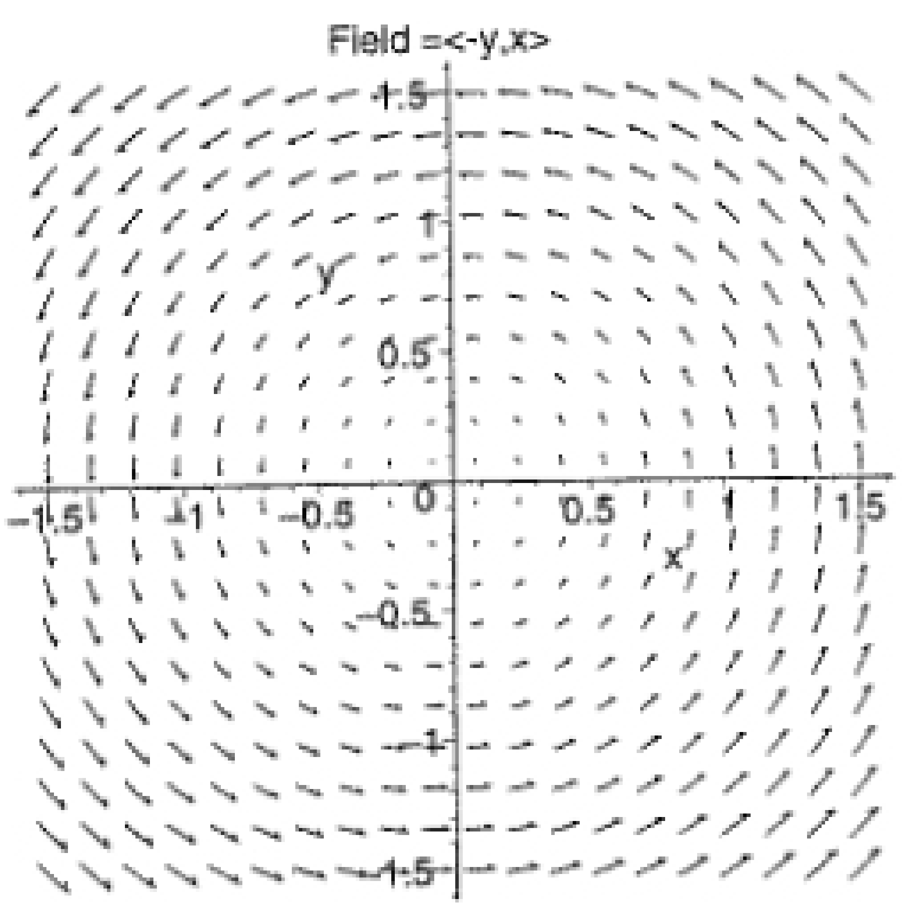
Math 2210-4
Monday 4/11

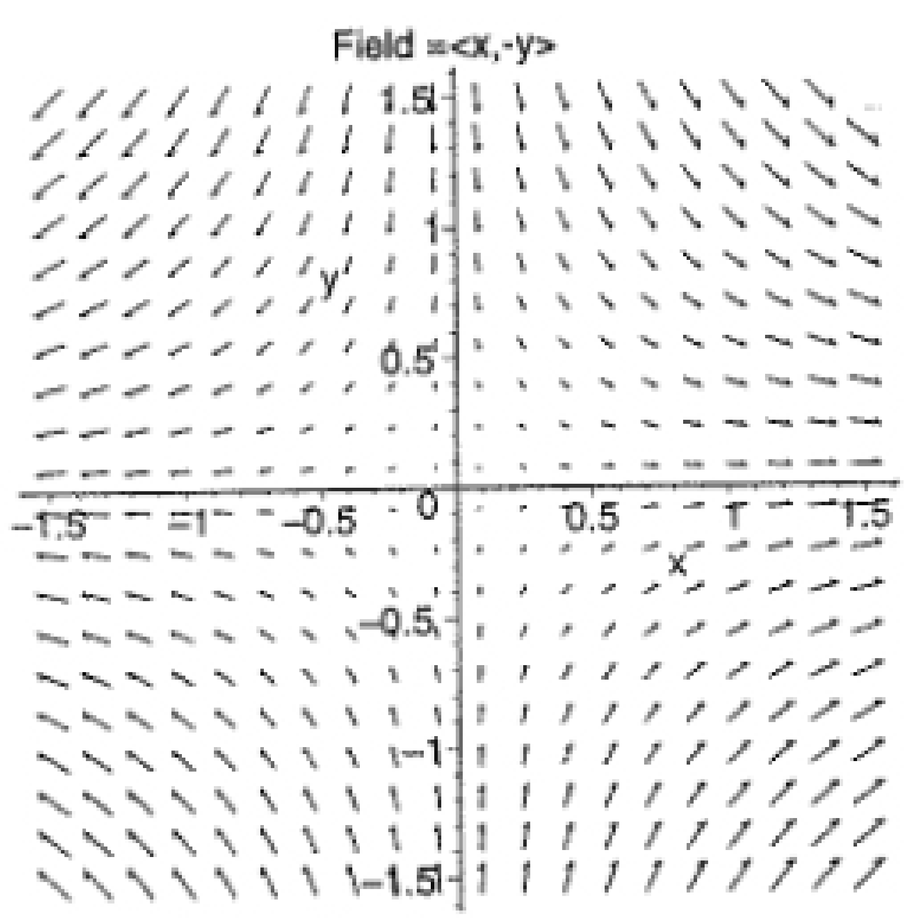
§17.1 Vector fields:
these arise naturally in physics as force fields, also as velocity fields in fluid motion



$\vec{F} = \langle M(x,y), N(x,y)\rangle = \langle x, y\rangle$
gradient field? ($\vec{F} = \nabla f$)



$\langle M, N\rangle = \langle -y, x\rangle$
gradient field?
(Think Escher!)



$\langle M, N\rangle = \langle x, -y\rangle$
gradient field?

Think of ∇ meaning $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle = \langle \partial_x, \partial_y, \partial_z \rangle$ in $\mathbb{R}^3$

$\boxed{\text{divergence}}$ or $\langle \partial_x, \partial_y \rangle$ in $\mathbb{R}^2$

$\downarrow$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle = M_x + N_y + P_z \quad \text{in } \mathbb{R}^3$$
$$= \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \rangle \cdot \langle M, N \rangle = M_x + N_y \quad \text{in } \mathbb{R}^3$$

$\boxed{\text{curl}}$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \quad \text{in } \mathbb{R}^3$$

in $\mathbb{R}^2$, scalar curl ($\text{curl}\,\vec{F}$) is $N_x - M_y$

<u>note</u> if $\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle$, then $\text{curl}\,\vec{F} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle = \vec{0}$!

i.e. $\nabla \times (\nabla f) = 0$!

Compute div, curl for our 3 $\mathbb{R}^2$ vector fields. Think about what they may mean. (in relation to the English meaning of <u>divergence</u> and <u>curl</u>)

So, $\text{curl}\,\vec{F}$ measures whether $\vec{F}$ could be a gradient vector field.
It <u>also</u> measures "rotation" in a vector field,
i.e. suppose the vector field is a field of tangent vectors for moving particles, as in a fluid flow.
It turns out $\text{curl}\,\vec{F}$ is a vector which measures how much they're rotating.

e.g. $\vec{F} = \langle -y, x, 0 \rangle$

$\vec{\nabla} \times \vec{F} =$

$\text{div}\,\vec{F}$ measures whether the vector field illustrates expansion or contraction

Chapter 17 is largely about making these heuristic descriptions precise, and is part of the mathematical foundation for many areas of physics

Example

$\vec{F} = \langle 2x+y, e^x, 2z \rangle$

Compute

$\vec{\nabla} \times \vec{F}$

$\vec{\nabla} \cdot \vec{F}$

Example in $\mathbb{R}^3$, $f(x,y,z) = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$ $\left(= -\frac{1}{e}\right)$.

Compute $-\nabla f = \vec{F}$

notice this is the inverse square law (of gravity & ~~electrostatics~~ electric charge).

Thus $\vec{\nabla} \times \vec{F} = \vec{0}$.

Show also $\vec{\nabla} \cdot \vec{F} = 0$ (in HW).