Math 231 Worksheet #5

1. Right Hand Approximation

Let $R_n f$ be the Right Hand Approximation to $\int_a^b f(x)\,dx$ over n equal partitions.
(a) Give a formula for $R_2 f$, $R_3 f$, and $R_n f$.
Solution:

$$R_2 f = f\left(a + \frac{b-a}{2}\right) \cdot \frac{b-a}{2} + f(b) \cdot \frac{b-a}{2}$$
$$R_3 f = f\left(a + \frac{b-a}{3}\right) \cdot \frac{b-a}{3} + f\left(a + 2\frac{b-a}{3}\right) \cdot \frac{b-a}{3} + f\left(b\right) \cdot \frac{b-a}{3}$$
$$R_n f = \left[f\left(a + \frac{b-a}{n}\right) + f\left(a + \frac{2}{n}(b-a)\right) + \ldots + f\left(a + \frac{n}{n}(b-a)\right)\right] \frac{b-a}{n}$$
$$= \sum_{i=1}^{n} f\left(a + \left(\frac{b-a}{n}\right) i\right) \cdot \frac{b-a}{n} = \sum_{i=1}^{n} f\left(a + i \cdot \Delta x\right) \Delta x \quad \text{with } \Delta x = \frac{b-a}{n}$$

(b) Explain why it is that if $|f'(x)| \leq K_1$ for all x in $[a,b]$, then for all x in $[a,b]$

$$|f(x) - f(a)| \leq K_1(x-a)$$

Solution:
By the Mean Value Theorem, there exists a c in (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
There exists a $K_1 > 0$ such that $|f'(x)| \leq K_1$ for all x in $[a,b]$.
Then $|f(x) - f(a)| \leq K_1(x-a)$.

(c) Use your result from (b) to show that if $|f'(x)| \leq K_1$ for all x in $[a,b]$ then $R_1 f$ differs from $\int_a^b f(x)\,dx$ by at most $\frac{K_1(b-a)^2}{2}$.
Solution:

Difference between integral and approximation:
$D = \int_a^b f(x)\,dx - f(b)\cdot(b-a) = \int_a^b f(x)\,dx - \int_a^b f(b)\,dx = \int_a^b \left(f(x) - f(b)\right)\,dx$
By MVT: $f(x) - f(b) = f'(c)(x-b)$ for some c in (b,x)
The derivative is assumed bounded: $|f'(x)| \leq K_1$ for all $x \in [a,b]$
Then:

$$\left|\int_a^b (f(x) - f(b))\,dx\right| \leq \int_a^b |f(x) - f(b)|\,dx = \int_a^b |f'(c)\cdot(x-b)| \leq \int_a^b K_1 \cdot |x-b|\,dx$$
$$= \int_a^b K_1 \cdot (b-x)\,dx$$

Then:
$\int_a^b K_1(b-x)\,dx = \left[-K_1\frac{(b-x)^2}{2}\right]_a^b = \left(-K_1\frac{(b-b)^2}{2}\right) - \left(-K_1\frac{(b-a)^2}{2}\right) = K_1\frac{(b-a)^2}{2}$

(d) Use your result from part (c) and your formula in part (a) to show that if $|f'(x)| \leq K_1$ for all x in $[a,b]$ then $R_n f$ approximates $\int_a^b f(x)\,dx$ to an error no more than $\frac{K_1(b-a)^2}{2n}$.

Solution:

The definition of R_n is to subdivide the interval $[a,b]$ into n equal pieces. If we label the points as $a = x_0 < x_1 < \cdots < x_n = b$ then $R_n[a,b] = \sum_{i=1}^n R_1[x_{i-1}, x_i]$. We also have $\int_a^b f(x)\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx$. Thus, by the triangle inequality we have that

$$\left|\int_a^b f(x)\,dx - R_n[a,b]\right| = \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^n R_1[x_{i-1}, x_i]\right| \leq \sum_{i=1}^n \left|\int_{x_{i-1}}^{x_i} f(x)\,dx - R_1[x_{i-1}, x_i]\right|$$

By part (c), we know that

$$\left|\int_{x_{i-1}}^{x_i} f(x)\,dx - R_1[x_{i-1}, x_i]\right| \leq \frac{K_1}{2}(x_i - x_{i-1})^2$$

and since we cut the interval $[a,b]$ into n equal pieces, each $(x_i - x_{i-1}) = \frac{b-a}{n}$ and so

$$\sum_{i=1}^n \left|\int_{x_{i-1}}^{x_i} f(x)\,dx - R_1[x_{i-1}, x_i]\right| \leq \sum_{i=1}^n \frac{K_1}{2}\left(\frac{b-a}{n}\right)^2 = n \cdot \frac{K_1}{2}\frac{(b-a)^2}{n^2} = \frac{K_1}{2}\frac{(b-a)^2}{n}$$

2. The Midpoint Approximation M_n can be used to approximate $\int_a^b f(x)\,dx$. If $|f^{(2)}(x)| \leq K_2$ for all x on $[a,b]$ then M_1 approximates $\int_a^b f(x)\,dx$ to an error no more than $K_2\frac{(b-a)^3}{24}$ (this is shown using the Mean Value Theorem). Use this to show more generally that when $|f^{(2)}(x)| \leq K_2$ for all x on $[a,b]$ then

$$\left|\int_a^b f(x)\,dx - M_n\right| \leq K_2\frac{(b-a)^3}{24n^2}.$$

Solution:

Following the argument from 2(d):

$$\begin{aligned}
\left|\int_a^b f(x)\,dx - M_n[a,b]\right| &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^n M_1[x_{i-1},x_i]\right| \\
\text{(triangle inequality)} \quad &\le \sum_{i=1}^n \left|\int_{x_{i-1}}^{x_i} f(x)\,dx - M_1[x_{i-1},x_i]\right| \\
\text{(error estimation)} \quad &\le \sum_{i=1}^n K_2 \frac{(x_i - x_{i-1})^3}{24} \\
\text{(equal partitions)} \quad &= \sum_{i=1}^n K_2 \frac{\left(\frac{b-a}{n}\right)^3}{24} \\
&= n \cdot K_2 \frac{(b-a)^3}{24n^3} = K_2 \frac{(b-a)^3}{24n^2}
\end{aligned}$$

3. Mary has a "fast process" to approximate $\int_a^b f(x)\,dx$ which she calls $P(f)$. She knows that if $|f^{(6)}(x)| \le K_6$ for all x in $[a,b]$ then $P(f)$ approximates $\int_a^b f(x)\,dx$ to an error no more than $\frac{K_6(b-a)^7}{48}$. Mary wants to use her process to numerically approximate an integral by subdividing an interval into n equal pieces and applying $P(f)$ to each of the smaller intervals and then adding up the result (much like the Left/Right/Midpoint rules do). If Mary calls her approximation $P_n f$, what is an upper bound for the error of $P_n f$ to approximate $\int_a^b f(x)\,dx$? Why?

   Solution:

   Following the solutions to 1(d) and 2:

$$\begin{aligned}
\left|\int_a^b f(x)\,dx - P_n[a,b]\right| &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx - \sum_{i=1}^n P_1[x_{i-1},x_i]\right| \\
\text{(triangle inequality)} \quad &\le \sum_{i=1}^n \left|\int_{x_{i-1}}^{x_i} f(x)\,dx - P_1[x_{i-1},x_i]\right| \\
\text{(error estimation)} \quad &\le \sum_{i=1}^n K_6 \frac{(x_i - x_{i-1})^7}{48} \\
\text{(equal partitions)} \quad &= \sum_{i=1}^n K_6 \frac{\left(\frac{b-a}{n}\right)^7}{48} \\
&= n \cdot K_6 \frac{(b-a)^7}{48n^7} = K_6 \frac{(b-a)^7}{48n^6}
\end{aligned}$$

   and so an upper bound for the error is $K_6 \frac{(b-a)^7}{48n^6}$.