Name KEY

Differential Equations and Linear Algebra 2250-1

Final Exam 8:00am 4 May 2006

Ch3. (Linear Systems and Matrices)

☐ [50%] Ch3(a): Find the **fourth entry on the third row** of the inverse matrix B^{-1} by the formula $B^{-1} = \mathbf{adj}(B)/\det(B)$. Evaluate determinants by any method: triangular, swap, combo, multiply, cofactor. The use of the 2×2 Sarrus' rule is expected.

$$B = \begin{bmatrix} 1 & 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ -1 & 0 & 1 & 3 & 4 \end{bmatrix}$$

☐ [25%] Ch3(b): Determine all values of k such that the system $R\mathbf{x} = \mathbf{f}$ has no solution.

$$R = \begin{bmatrix} 2 & 1 & k \\ 2 & -k & -2 \\ 0 & 0 & k \end{bmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

☐ [25%] Ch3(c): Let A be a 3×3 triangular matrix with diagonal entries $7, -11, 1$. Prove that $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.

☐ [25%] Ch3(d): Let A denote a 3×4 matrix. Explain from theory why $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

☐ [25%] Ch3(e): Infinitely many 3×3 matrices A exist such that A^3 is the zero matrix but A^2 is not the zero matrix. Display one such matrix A and justify the claim.

Ch3(a) $\mathrm{Cof}(B,4,3) = (-1)^{4+3}\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & 4 \end{vmatrix} = 0$. B^{-1} entry $= \dfrac{\text{cofactor}}{\text{determinant}} = \boxed{0}$

$\det(B) = 24$

Ch3(b) Sequence to rref stops at $\left(\begin{array}{ccc|c} 2 & 1 & k & 0 \\ 0 & -k-1 & -2-k & 1 \\ 0 & 0 & k & 0 \end{array}\right)$. If $-k-1=0$, Then $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$ give a signal eq after combo. Otherwise, a sol always exists.

answer: $\boxed{\text{No sol } k=-1}$

Ch3(c) Then $\det(A) = 7(-11)(1) = -77 \neq 0$, So A^{-1} exists and $\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$.

Ch3(d) rank + nullity = 4, Only 3 rows $\Rightarrow$ rank ≤ 3 $\Rightarrow$ nullity ≥ 1 $\Rightarrow$ one free var. Therefore, at least one free var implies ∞-many solutions.

Ch3(e) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

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Ch4. (Vector Spaces)

☐ [40%] Ch4(a): State an RREF test (not a determinant test) to detect the independence or dependence of fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v3}$ in $\mathcal{R}^4$ [10%]. Apply the test to the vectors below [25%]. Report **independent** or **dependent** [5%].

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

☐ [60%] Ch4(b): Define V to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^4$ such that $x_1 + x_4 = 0$ and $\mathbf{c} \cdot \mathbf{x} = 0$, where $\mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 3 \end{pmatrix}$. Prove that V is a subspace of $\mathcal{R}^4$.

☐ [60%] Ch4(c): Find a basis of fixed vectors in $\mathcal{R}^4$ for (1) the column space of the 4×4 matrix A below [30%] and (2) the row space of the 4×4 matrix A below [30%]. The two displayed bases **must** consist of columns of A and rows of A, respectively.

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -2 & -1 \\ 2 & -2 & -1 & -1 \\ 3 & -3 & 0 & -1 \end{pmatrix}$$

☐ [40%] Ch4(d): Find a 4×4 system of linear equations for the constants a, b, c, d in the partial fractions decomposition below [10%]. Solve for a, b, c, d, showing all **RREF** steps [25%]. Report the answers [5%].

$$\frac{4x^2 - 12x + 4}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

Ch4(a) Test: independent $\iff$ rank(aug(v_1, v_2, v_3)) = 3. Apply: $\boxed{\text{independent.}}$

Ch4(b) $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The restriction eqs are $A\vec{x} = \vec{0}$. Apply Theorem 2 of E&P.

Ch4(c) Stop in rref sequence at $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then col 1, 3 = pivots = indep cols of A.
Repeat with A^T to get rows 1, 2 = indep rows of A

Ch4(d) Heaviside coverup gives $\boxed{b=-1}$ and $\boxed{d=5}$. Clear fractions, then substitute $x=0, x=2$ to get a 2×2 system for a, c. Solve it. $\boxed{a=0}$ $\boxed{c=0}$

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Ch5. (Linear Equations of Higher Order)

☐ [25%] Ch5(a): Using the *recipe* for higher order constant-coefficient differential equations, write out the general solutions of the differential equations whose characteristic equations are given below.

$$\textbf{1}.[12\%] \quad r^4(r^2-5r)^2(r^2-25)=0,$$
$$\textbf{2}.[13\%] \quad (r+4)^2(r^2+2r+2)^3(r^2-16)^2=0$$

☐ [25%] Ch5(b): Given a damped spring-mass system $mx''(t)+cx'(t)+kx(t)=0$ with $m=15$, $c=17$ and $k=4$, solve the differential equation [25%] and classify the answer as over-damped, critically damped or under-damped [5%].

☐ [50%] Ch5(c): Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

$$y^{iv}+4y''=x^2(1+2e^{2x})+3x\sin 2x+2\sin x\cos x$$

☐ [25%] Ch5(d): Find the steady-state periodic solution for the equation

$$x''+4x'+20x=3\cos(2t).$$

Ch5(a) 1. roots $= 0,0,0,0,0,0,5,5,5,-5$. $y=u_1e^{0x}+u_2e^{5x}+u_3e^{-5x}$,
$u_1=c_1+c_2x+c_3x^2+c_4x^3+c_5x^4+c_6x^5$, $u_2=c_7+c_8x+c_9x^2$, $u_3=c_{10}$

2. roots $= -4,-4,-4,-4,4,4,-1\pm i,-1\pm i,-1\pm i$ (12 roots)
$y=u_1e^{-4x}+u_2e^{4x}+u_3e^{-x}\cos x+u_4e^{-x}\sin x$, $u_1=c_1+c_2x+c_3x^2+c_4x^3$,
$u_2=c_5+c_6x$, $u_3=c_7+c_8x+c_9x^2$, $u_4=c_{10}+c_{11}x+c_{12}x^2$

Ch5(b) $(5r+4)(3r+1)=0$ has roots $-4/5,-1/3$. overdamped. Sol is
$x(t)=c_1e^{-4t/5}+c_2e^{-t/3}$

Ch5(c) $r^2(r^2+4)=0$ homogeneous roots $=0,0,\pm 2i$. Since $2\sin x\cos x=\sin 2x$,
the corrected trial sol is $y=y_1+y_2+y_3$, where $y_1=x^2(d_1+d_2x+d_3x^2)$,
$y_2=(d_4+d_5x+d_6x^2)e^{2x}$, $y_3=x([d_7+d_8x]\cos 2x+[d_9+d_{10}x]\sin 2x)$

Ch5(d) Trial Sol $x=d_1\cos 2t+d_2\sin 2t$ gives eqs $\begin{cases}16d_1+8d_2=3\\-8d_1+16d_2=0\end{cases}$
Solving, $d_1=3/20$, $d_2=3/40$

Cramer's rule
$\Delta=320$
$\Delta_1=(16)(3)$
$\Delta_2=(8)(3)$

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