Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. (The actual exam will have more free space). If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T	F	Suppose A is an $m \times n$ matrix, where $n < m$. If the rank of A is m, then there is a vector $y \in \mathbf{R}^m$ for which the system $Ax = y$ has no solutions.
T	F	The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ is invertible.
T	F	The rank of an lower-triangular matrix equals the number of non-zero entries along the diagonal.
T	F	The row reduced echelon form of a 3×3 matrix of rank 2 is one of the following $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
T	F	The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a shear.
T	F	For any matrix A, one has $\dim(\ker(A)) = \dim(\ker(\mathrm{rref}(A)))$.
T	F	If $\ker(A)$ is included in $\mathrm{im}(A)$, then A is not invertible.
T	F	There exists an invertible 3×3 matrix, for which 7 of the 9 entries are π.
T	F	The dimension of the image of a matrix A is equal to the dimension of the image of the matrix $\mathrm{rref}(A)$.
T	F	There exists an invertible $n \times n$ matrix whose inverse has rank $n-1$.
T	F	If A and B are $n \times n$ matrices, then AB is invertible if and only if both A and B are invertible.
T	F	There exist matrices A, B such that A has rank 4 and B has rank 7 and AB has rank 5.
T	F	There exist matrices A, B such that A has rank 2 and B has rank 7 and AB has rank 1.
T	F	If for an invertible matrix A one has $A^2 = A$, then $A = 1$.
T	F	If an invertible matrix A satisfies $A^2 = 1$, then $A = 1$ or $A = -1$.
T	F	The matrix $\begin{bmatrix} c-1 & -1 \\ 2 & c+1 \end{bmatrix}$ is invertible for every real number c.
T	F	For 2×2 matrices A and B, if $AB = 0$, then either $A = 0$ or $B = 0$.
T	F	The determinant of a shear in the plane is always 1.
T	F	The plane $x + y - z = 1$ is a linear subspace of three dimensional space.
T	F	If T is a rotation in space around an angle $\pi/6$ around the z axes, then the linear transformation $S(x) = T(x) - x$ is invertible.

	Total

Problem 2) (10 points)

Determine for each of the following matrices A, whether the system $A\vec{x} = \vec{e}_1$ has zero, one or infinitely many solutions and find the dimension of the image of A in each case:

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

c) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$.

d) $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

e) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

f) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

g) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Problem 3) (10 points)

a) Write the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as a product of a rotation and a dilation.

b) What is the length of the vector $\vec{v} = A^{100}e_1$, where e_1 is the first basis vector?

c) In which direction does the vector $\vec{v}$ point?

d) Find a matrix B such that $B^2 = A$.

Problem 4) (10 points)

Let A be a 3×3 matrix such that $A^2 = 0$. That is, the product of A with itself is the zero matrix.

a) Verify that $\mathrm{Im}(A)$ is a subspace of $\ker(A)$.