Math 2280-2, Practice Midterm Exam 1

February 12, 2008

Total: 110/100 points

Problem 1 (25 pts) A certain population P can be modeled by the differential equation

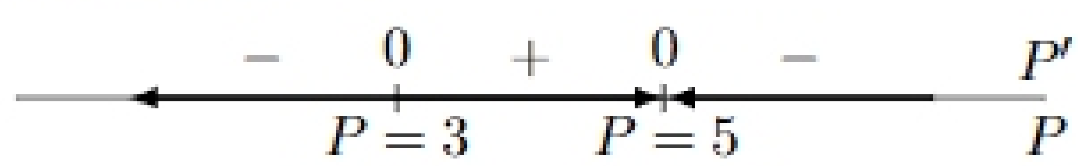
$$\frac{dP}{dt} = -P^2 + 8P - 15$$

(a) (4 pts) Find the equilibrium solutions of the differential equation

We have $-P^2 + 8P - 15 = -(P-3)(P-5)$. Thus the equilibrium solutions of the differential equation are $P = 3$ and $P = 5$.

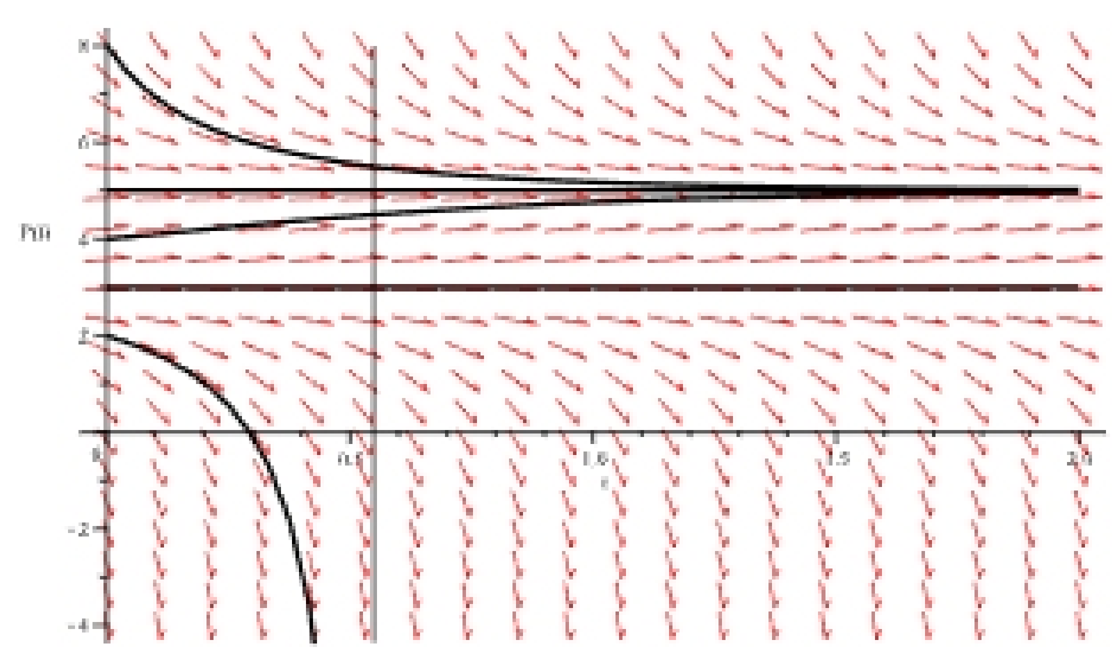
(b) (4 pts) Sketch the phase diagram for the differential equation. Identify the type of equilibrium solution.

The phase diagram for this DE is,



We see that $P = 3$ is unstable and $P = 5$ is stable.

(c) (4 pts) Sketch the slope field for the differential equation together with the equilibrium solutions and a few representative solutions (**note:** you do not need to find the formulas of the solutions to sketch them).



(d) (9 pts) Solve the differential equation with initial condition $P(0) = 2$. (Here you do need to find the formula for $P(t)$ for this initial condition).

We rewrite the differential as $dP/dt = -(P-3)(P-5)$ and use separation of variables:

$$\int \frac{dP}{(P-3)(P-5)} = -\int dt$$
$$\Rightarrow \frac{1}{2}\int \left(\frac{-1}{P-3} + \frac{1}{P-5}\right) dP = -t + C_1$$
$$\Rightarrow \ln\left|\frac{P-5}{P-3}\right| = -2t + C_2$$
$$\Rightarrow \frac{P-5}{P-3} = C_3 e^{-2t} \quad \text{since } P(0) = 2 \;\Rightarrow\; \frac{P-5}{P-3} > 0 \text{ for } t \text{ near zero.}$$

Using the initial condition in the last equation we get $C_3 = 3$. Solving for P gives,

$$P(t) = \frac{9e^{-2t} - 5}{3e^{-2t} - 1}.$$

This is plotted in the slope field above.

(e) (4 pts) Check that your solution is consistent with the slope field, that is study the behavior of $P(t)$. In particular, find the value T for which $\lim_{t \to T} P(t) = -\infty$.

The denominator of $P(t)$ can be zero. This happens for T such that $3e^{-2T} = 1$, i.e. for $T = \ln(3)/2$. This is the vertical asymptote plotted in the slope field and is consistent with the behavior of $P(t)$.

Problem 2 (25 pts) Consider a 10 gallon tank, initially full of pure water. A brine with concentration of 1 pound of salt per gallon enters the tank at the rate of 1 gallon per minute. The mixture at the tank is kept perfectly mixed, and the tank has a hole that lets the solution escape at 2 gallons per minute. Thus the tank will be empty after exactly 10 minutes and the volume of solution in the tank is $V(t) = 10 - t$.

(a) (5 pts) Show that the quantity of salt $x(t)$ (pounds) inside the tank before it is empty satisfies the differential equation,

$$\frac{dx}{dt} = 1 - \frac{2}{10-t}x.$$

Using the notation from class we have that $r_i = 1$, $c_i = 1$ and $r_o = 2$. The concentration of salt in the tank is $c_o = x/V = x/(10-t)$. Thus at any time $0 \le t \le 10$, the rate at which the quantity of salt in the tank changes is

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 1 - \frac{2}{10-t}x.$$

(b) (15 pts) Use the integrating factor method to solve the above differential equation.

We rewrite the differential equation as,

$$\frac{dx}{dt} + \frac{2}{10-t}x = 1.$$

The integrating factor method tells us to multiply on both sides of the DE by

$$e^{\int 2dt/(10-t)} = e^{-2\ln(10-t)} = \frac{1}{(10-t)^2}, \quad \text{(Note the minus sign in the exponential)}$$

to obtain

$$\frac{d}{dt}\left(\frac{x}{(10-t)^2}\right) = \frac{dx}{dt}\frac{1}{(10-t)^2} + \frac{2}{(10-t)^3}x = \frac{1}{(10-t)^2},$$

which by integration gives

$$\frac{x(t)}{(10-t)^2} = \int \frac{dt}{(10-t)^2} + C_1 = \frac{1}{10-t} + C_1.$$

Thus the general form for x is,

$$x(t) = (10-t) + C_1(10-t)^2.$$

Since there is no salt in the tank at $t = 0$ we get the value of the constant C_1

$$0 = x(0) = 10 + 100C_1 \quad \Rightarrow C_1 = -1/10.$$

and thus the answer

$$x(t) = (10-t) - \frac{1}{10}(10-t)^2.$$

(c) (5 pts) What is the maximum quantity of salt ever in the tank?

The maximum amount salt occurs at T where $x'(T) = 0$. Using the expression we have found for $x(t)$ we get

$$x'(t) = -1 + \frac{1}{5}(10-t).$$

Setting $x'(T) = 0$ we get $T = 5$. Evaluating, the maximum quantity of salt in the tank is $x(T) = 5/2 = 2.5$ pounds.