Section 3: The Multivariate Normal

20 Sept. 01

Properties of the normal.

Definition of the multivariate normal, including singular case

$Z_1, \ldots, Z_\ell$: $I\,N(0,1)$

$$\underset{\ell\times 1}{\underset{\sim}{Z}} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_\ell \end{bmatrix}$$

$$E\,\underset{\sim}{Z} = 0, \qquad \operatorname{var} \underset{\sim}{Z} = \underset{\sim}{\Sigma}_{ZZ} = \underset{\sim}{I}$$

$$\underset{m\times 1}{\underset{\sim}{Y}} = \underset{m\times 1}{\underset{\sim}{\alpha}} + \underset{m\times \ell}{\underset{\sim}{\beta}}\,\underset{\ell\times 1}{\underset{\sim}{Z}} \quad \rightleftharpoons \quad N_m(\underset{\sim}{\mu}_Y, \underset{\sim}{\Sigma}_{YY})$$

$$\underset{\sim}{\mu}_Y = \underset{\sim}{\alpha}, \qquad \underset{\sim}{\Sigma}_{YY} = \underset{\sim}{\beta}\,\underset{\sim}{\beta}'$$

<u>Theorem</u>. $\underset{n\times 1}{\underset{\sim}{X}} = \underset{n\times 1}{\underset{\sim}{A}} + \underset{n\times m}{\underset{\sim}{B}}\,\underset{m\times 1}{\underset{\sim}{Y}}$ is $N_n(\underset{\sim}{A} + \underset{\sim}{B}\,\underset{\sim}{\mu}_Y, \underset{\sim}{B}\,\underset{\sim}{\Sigma}_{YY}\,\underset{\sim}{B}')$

<u>Proof</u>. Linear combination of Z's.

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Theorem. The m.g.f.

$$E\{e^{\underset{\sim}{\theta}' \underset{\sim}{Y}}\} = e^{\underset{\sim}{\theta}' \underset{\sim}{\mu} + \frac{1}{2} \underset{\sim}{\theta}' \underset{\sim}{\Sigma} \underset{\sim}{\theta}}$$

Proof.

$$\underset{\sim}{\theta}' \underset{\sim}{Y} = \underset{\sim}{\theta}' \underset{\sim}{\alpha} + \underset{\sim}{\theta}' \underset{\sim}{\beta} \underset{\sim}{Z} = \underset{\sim}{\theta}' \alpha + \underset{\sim}{\delta}' \underset{\sim}{Z} \qquad \underset{\sim}{\delta} = \underset{\sim}{\beta}' \underset{\sim}{\theta}$$

$$= \gamma + \sum_i \delta_i Z_i$$

$$E\, e^{\underset{\sim}{\theta}' \underset{\sim}{Y}} = e^{\gamma} e^{\frac{1}{2} \sum \delta_i^2} = e^{\gamma} e^{\frac{1}{2} \underset{\sim}{\delta}' \underset{\sim}{\delta}}$$

$$= \exp\{\underset{\sim}{\theta}' \underset{\sim}{\alpha} + \frac{1}{2} \underset{\sim}{\theta}' \underset{\sim}{\beta} \underset{\sim}{\beta}' \underset{\sim}{\theta}\}$$

Shows well defined wrt choices of β
- need $\underset{\sim}{\beta} \underset{\sim}{\beta}'$ unique

Corollary. $\underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{\Sigma}_{22} \end{bmatrix} \Rightarrow \underset{\sim}{Y}_1 \perp\!\!\!\perp \underset{\sim}{Y}_2$

$$\underset{\sim}{\theta}' \underset{\sim}{\Sigma} \underset{\sim}{\theta} = \underset{\sim}{\theta}_1' \underset{\sim}{\Sigma}_{11} \underset{\sim}{\theta}_1 + \underset{\sim}{\theta}_2' \underset{\sim}{\Sigma}_{22} \underset{\sim}{\theta}_2$$

i.e. mgf factors