Smooth functions $f(x)$ on $[-\pi, \pi]$ form a linear space X. There is an **inner product** in X defined by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$$

It allows to define angles, length, projections in X as we did in finite dimensions.

THE FOURIER BASIS.

THEOREM. The functions $\{\cos(nx), \sin(nx), 1/\sqrt{2}\}$ form an orthonormal set in X.

Proof. To check linear independence a few integrals need to be computed. For all $n, m \geq 1$, with $n \neq m$ you have to show:

$$\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle = 1$$
$$\langle \cos(nx), \cos(nx)\rangle = 1, \langle \cos(nx), \cos(mx)\rangle = 0$$
$$\langle \sin(nx), \sin(nx)\rangle = 1, \langle \sin(nx), \sin(mx)\rangle = 0$$
$$\langle \sin(nx), \cos(mx)\rangle = 0$$
$$\langle \sin(nx), 1/\sqrt{2}\rangle = 0$$
$$\langle \cos(nx), 1/\sqrt{2}\rangle = 0$$

To verify the above integrals in the homework, the following trigonometric identities are useful:

$$\begin{aligned} 2\cos(nx)\cos(my) &= \cos(nx-my)+\cos(nx+my) \\ 2\sin(nx)\sin(my) &= \cos(nx-my)-\cos(nx+my) \\ 2\sin(nx)\cos(my) &= \sin(nx+my)+\sin(nx-my) \end{aligned}$$

FOURIER COEFFICIENTS. The **Fourier coefficients** of a function f in X are defined as

$$a_0 = \langle f, 1/\sqrt{2}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\,dx$$
$$a_n = \langle f, \cos(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$
$$b_n = \langle f, \sin(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

FOURIER SERIES. The **Fourier representation** of a smooth function f is the identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$$

We take it for granted that the series converges and that the identity holds for all x where f is continuous.

ODD AND EVEN FUNCTIONS. Here is some advise which can save time when computing Fourier series:

If f is odd: $f(x) = -f(-x)$, then f has a sin series.
If f is even: $f(x) = f(-x)$, then f has a cos series.
If you integrate an odd function over $[-\pi, \pi]$ you get 0.
The product of two odd functions is even, the product between an even and an odd function is odd.

EXAMPLE 1. Let $f(x) = x$ on $[-\pi, \pi]$. This is an odd function $(f(-x)+f(x) = 0)$ so that it has a sin series: with $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{-1}{\pi}(x\cos(nx)/n + \sin(nx)/n^2|_{-\pi}^{\pi}) = 2(-1)^{n+1}/n$, we get $x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin(nx)$. For example

$$\frac{\pi}{2} = 2(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}...)$$

is a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7\cos(5x)$. This **trigonometric polynomial** is already the Fourier series. There is no need to compute the integrals. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos(nx)\, dx = \frac{\sin(nx)}{\pi n}\Big|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m+1$ is odd and 0 else. So, the series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - ...\right)$$

Remark. The function in Example 3 is not smooth, but Fourier theory still works. What happens at the discontinuity point $\pi/2$? The Fourier series converges to 0. Diplomatically it has chosen the point in the middle of the limits from the right and the limit from the left.

FOURIER APPROXIMATION. For a smooth function f, the Fourier series of f converges to f. The Fourier coefficients are the coordinates of f in the Fourier basis.

The function $f_n(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$ is called a **Fourier approximation** of f. The picture to the right plots a few approximations in the case of a piecewise continuous even function given in example 3).

FUNCTIONS OF TWO VARIABLES. We consider functions $f(x,t)$ which are for fixed t a piecewise smooth function in x. Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the motion of a **function** f in time t. While the governing equation for a vector was an ordinary differential equation $\dot{x} = Ax$ (ODE), the describing equation is now be a **partial differential equation** (PDE) $\dot{f} = T(f)$. The function $f(x,t)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t. The motion of these dynamical systems will be easy to describe in the orthonormal Fourier basis $1/\sqrt{2}, \sin(nx), \cos(nx)$ treated in an earlier lecture.

PARTIAL DERIVATIVES. We write $f_x(x,t)$ and $f_t(x,t)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(x,t)$ means that we differentiate twice with respect to x.

Example: for $f(x,t) = \cos(x+4t^2)$, we have
- $f_x(x,t) = -\sin(x+4t^2)$
- $f_t(x,t) = -8t\sin(x+4t^2)$.
- $f_{xx}(x,t) = -\cos(x+4t^2)$.

One also uses the notation $\frac{\partial f(x,y)}{\partial x}$ for the partial derivative with respect to x. Tired of all the "partial derivative signs", we always write $f_x(x,t)$ for the partial derivative with respect to x and $f_t(x,t)$ for the partial derivative with respect to t.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x,t)$ in which different partial derivatives occur.

- $f_t(x,t) + f_x(x,t) = 0$ with $f(x,0) = \sin(x)$ has a solution $f(x,t) = \sin(x-t)$.
- $f_{tt}(x,t) - f_{xx}(x,t) = 0$ with $f(x,0) = \sin(x)$ and $f_t(x,0) = 0$ has a solution $f(x,t) = (\sin(x-t) + \sin(x+t))/2$.

THE HEAT EQUATION. The temperature distribution $f(x,t)$ in a metal bar $[0,\pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t)$$

This partial differential equation tells that the rate of change of the temperature at x is proportional to the second space derivative of $f(x,t)$ at x. The function $f(x,t)$ is assumed to be zero at both ends of the bar and $f(x) = f(x,t)$ is a given initial temperature distribution. The constant μ depends on the heat conductivity properties of the material. Metals for example conduct heat well and would lead to a large μ.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d}{dt} f = \mu D^2 f$$

We will solve the problem in the same way as we solved linear differential equations:

$$\frac{d}{dt} \vec{x} = A\vec{x}$$

where A is a matrix - by diagonalization.

We use that the Fourier basis is just the diagonalization: $D^2 \cos(nx) = -n^2 \cos(nx)$ and $D^2 \sin(nx) = -n^2 \sin(nx)$ show that $\cos(nx)$ and $\sin(nx)$ are eigenfunctions to D^2 with eigenvalue n^2. By a symmetry trick, we can focus on sin-series from now on.