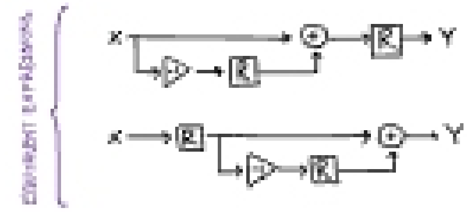
Lecture 4

Multiple Representations of Systems

Operator representations are compact and preserve flow information
- Combine best features of difference equations and block diagrams
- Provide a convenient way to reason out a system

Example: Y = (1 - R)X



Operator Expressions Obey Rules of Polynomial Algebra

Operator expressions are commutative and associative, and multiplication distributes over addition

Equivalent polynomials —> equivalent systems (when started at rest)

System Functionals

We can represent an entire system by its operator representation, which we call the system function

Representing a system by a system functional

Example one: Y = HX

Example two: Y = HX

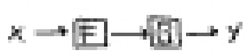
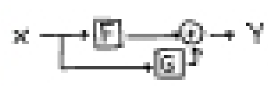
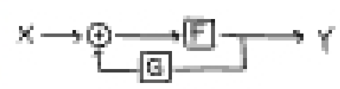
Functionals for systems without feedback are polynomials in R

General Framework for Thinking about Systems

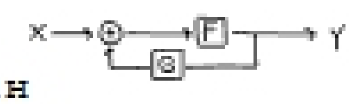
A system can be represented by a delay or gain or the combination of the two systems using a Cascade or FeedforwardAdd or FeedbackAdd
- Primitives
  - Delays
  - Gains
- Composition
  - Cascade
  - FeedforwardAdd
  - FeedbackAdd
- The rules of composition are recursive

Composing Complicated Systems

Any of the rules of composition can be added to any two systems to generate a new type of system

Cascade:  Feedforward:  Feedback: 

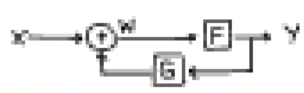
Check yourself: System Functionals

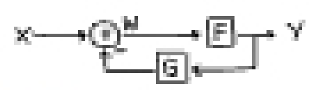


Correct answer: 1

Determine the system functional H

Black's Equation





Black's equation has two common forms

Difference between the two: the change of sign G

The second form is useful in most control applications where the goal is to make Y and X converse

Check yourself: Representing systems

Correct answer: 2

Assume that F and G can be represented by polynomials in R

How many of the following systems can be represented by a polynomial in R
- Cascade(F, G)
  - F x G is a polynomial
- FeedforwardAdd(F, G)
  - F + G is a polynomial
- FeedbackAdd(F, Gain(1))
  - F/(1 - F) is generally not a polynomial
- FeedbackAdd (Gain(1), G)
  - F/ (1 - FG)
- FeedbackAdd(F, G)

Check yourself: Representing systems

Correct answer: 5

Assume that each of F and G can be represented by a ratio of two polynomials in R

How many of the following systems can be represented by a ratio of two polynomials?
- Cascade(F, G)
- FeedforwardAdd(F, G)
- FeedbackAdd(F, Gain(1))
- FeedbackAdd (Gain(1), G)
- FeedbackAdd(F, G)

System Functionals as Abstraction

Systems that can be represented by a Delay or a gain or by the combination of two systems using Cascade, FeedforwardAdd, or FeedbackAdd can also be represented by a ratio of polynomials in R

R = 1/K and K = 1/R

Cascade: $\left(\frac{N_1}{D_1}, \frac{N_2}{D_2}\right) = \frac{N_1 N_2}{D_1 D_2}$ Feedforward: $\left(\frac{N_1}{D_1}, \frac{N_2}{D_2}\right) = \frac{N_1 D_2 + N_2 D_1}{D_1 D_2}$ Feedback: $\left(\frac{N_1}{D_1}, \frac{N_2}{D_2}\right) = \frac{N_1 D_2}{D_1 D_2 - N_1 N_2}$

System Functionals can be used interchangeably with primitives

PCAP Framework for Managing Complexity

A system can be represented by a delay or a gain or the combination of two systems using Cascade, FeedforwardAdd, or FeedbackAdd

- Primitives
  - Delays — R
  - Gains — K
- Composition
  - Cascade — FG
  - FeedforwardAdd — F+G
  - FeedbackAdd — F/(1 - FG)
- Abstraction
  - System Functional — ratio of polynomials in R

Poles

Ways to find the poles of a system

Write the system functional in special form

Write the operator expression in special form

Substitute R = 1/z and find the roots of the denominator

All the methods give the same result

Complex Poles

What if a pole has a non-zero imaginary part?

Example: $\frac{Y}{X} = \frac{1}{1 - R + R^2} = \frac{1}{1 - \frac{1}{z} + \frac{1}{z^2}} = \frac{z^2}{z^2 - z + 1}$

The poles are at $z = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, corresponding modes are $\left(\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\right)^n$

Powers of complex numbers are easy to compute using polar forms

Express the pole at $z = a + jb$ as $re^{j\theta}$ where $r^2 = a^2 + b^2$ and $\tan\theta = b/a$

Then the mode is $(re^{j\theta})^n = r^n e^{j\theta n}$

geometric growth of magnitude

linear growth of angle

Complex Roots

An isolated complex root can result only from a difference equation with complex-valued coefficients

Example: $\frac{Y}{X} = \frac{1}{1 - re^{j\Omega}R}$ Corresponding difference equation: $y[n] - re^{j\Omega}y[n-1] = x[n]$

Difference equations that represent physical systems have real valued coefficients

Difference equations with real-valued coefficients generate real-valued outputs from real-valued inputs

But systems with real valued coefficients can have complex-valued poles

The sum of the modes associate with complex conjugates is real

Check yourself: Complex Poles

Correct answer: 2

Output of a system with poles at $z = re^{\pm j\Omega}$

What are the values for r and Ω? $0.5 < r < 1$ and $\Omega \approx 0.5$

Designing a Control System

We have already built several feedback systems to control the robot:

- wallFinder—move forward or backward to position robot a desired distance from the wall in front of it
- wallFollower—move the robot parallel to a wall maintaining a desired distance from the wall

We can use the Signals and Systems Abstraction to gain insight into the general problem of designing a feedback controller

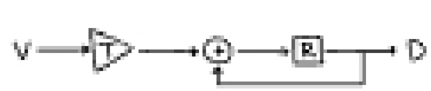
Controlling Accumulation

Both wallFinder and wallFollower contained accumulators to model integrative processes

- wallFinder: command forward velocity to set forward distance
- wallFollower: command rotational velocity to set lateral distance

Integrator: $V \longrightarrow \boxed{\int \partial t} \longrightarrow D$

Accumulator model: d[n+1] = d[n] + Tv[n]



Finding the Optimal Gain

Consider a proportional controller for the accumulator system

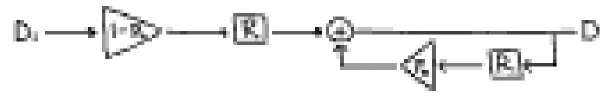
Replace accumulator with equivalent system functional

This gives us an equivalent system with a single block

Unit Sample Response

The system functional contains a single pole at $p_0 = 1 - KT$

$$\frac{D}{D_i} = \frac{KTR}{1-(1-KT)R} = \frac{(1-p_0)R}{1-p_0 R}$$



The numerator is just a gain and a delay

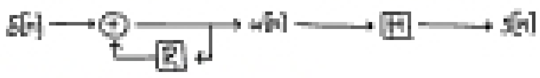
Step Response

We are often interested in the step response of a control system

Start with the output d[n] = 0 while the input di[n] is held constant where d[n] is the sonarDistance and di[n] is the desiredDistance

Calculating unit-step response

Unit Step response s[n] is the response of H to the unit-step signal u[n], which is constructed by accumulation of the unit sample δ[n]



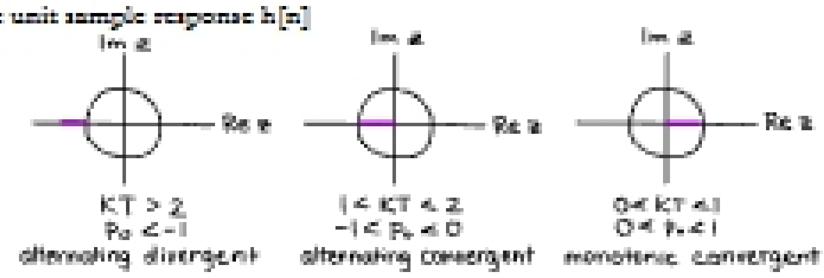
Commute and relabel signals: 

The unit step response s[n] is equal to the accumulated responses to the unit sample response h[n]

Root Locus

The poles of the system functional provide insight for choosing K

$$\frac{D}{D_i} = \frac{KTR}{1-(1-KT)R} = \frac{(1-p_0)R}{1-p_0 R} \longrightarrow p_0 = 1 - KT$$



Check yourself: Root Locus

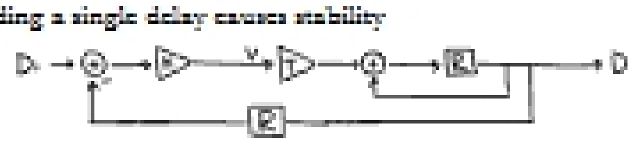
Correct answer: 2

Find KT for fastest convergence of unit-sample response. KT = 1

Destabilizing Effect of Delay

The optimum gain allows the proportional controller system to converge in a single step

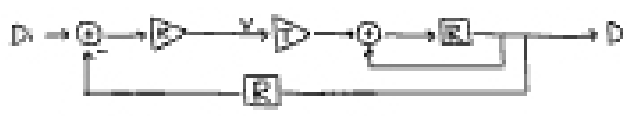
Adding a single delay causes stability



Check yourself: System Functional

Correct answer: 4

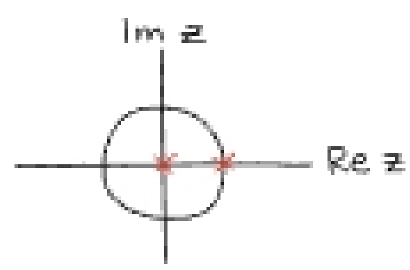
Find the system functional: $H = \frac{KTR}{1 - R + KTR^2}$



Feedback and Control: Poles

If KT is small, the poles at z ≈ KT and z ≈ 1 - KT

$$z = \frac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 - KT} = \frac{1}{2}\left(1 \pm \sqrt{1-4KT}\right) \approx \frac{1}{2}\left(1 \pm (1-2KT)\right) = 1-KT,\ KT$$



Pole near 0 generates fast response

Pole near 1 generates slow response

The slow mode dominates the response

Check yourself: Poles

Correct answer: 5

The period of oscillation is 6

Check yourself: Poles

Correct answer: 2

The fastest response is when KT = 1/4

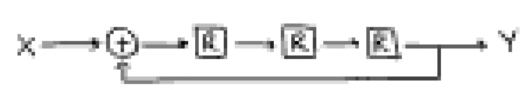
Destabilizing Effect of Delay

Adding delay tends to destabilize control systems

Check yourself: Possible Test Question

Correct answer:

How many of the following statements are true?



- This system has three poles
- Unit-sample response is the sum of three geometric sequences
- Unit-sample response is y[n] = 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...
- Unit-sample response is y[n] = 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...
- One of the poles is at z = 1

Designing Control Systems: Summary

System functionals provide a convenient summary of information that is important for designing control systems