Week 2

System Functionals

Sometimes we want to speak of the relationship between x and y without a particular x and y in mind

We can do this by defining a function *f*: $f(x) = x + 6$

We can do the same thing with LTI systems by defining *system functionals*

We will refer to Y/X as the *system functional*

It characterises the operation of a system, independent of the particular input and output signals involved

It is *functional* in the sense that it is itself an operation which can be applied to arbitrary input signals

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \cdots}{a_0 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \cdots}$$

Primitive Systems

We can specify system functions for each of our system primitives

A *gain* element is governed by operator equation **Y = kX**, for constant k, so its system functional is $\mathcal{H} = \frac{Y}{X} = k$

A *delay* element is governed by operator equation **Y = RX**, so its system functional is $\mathcal{H} = \frac{Y}{X} = R$.

Combining System Functionals

We have three basic composition operations: sum, cascade, and feedback

This PCAP system, as our previous ones have been , is *compositional*, in the sense that whenever we make a new system function out of existing ones, it is a system functional on its on right, which can be an element in further compositions

Example: $y[n] = x[n] + z[n]$ $z[n] = y[n-1] + z[n-1]$ → operator equations → $Y = X + Z$ $Z = RY + RZ$ → eliminate Z → $Z = \frac{R}{(1-R)}Y$ $Y = X + \frac{1}{1-R}Y$

Addition

The system functional of the *sum* of two systems is the sum of their systems

Cascade

The system functional of the cascade of two systems is the product of their system functionals

Feedback

There are several ways of connection systems in feedback

We study a particular case of *negative feedback* combination, which results in a classical formula called *Black's formula*

Predicting System Behavior

We will see how we can use properties of the system function to predict how the system will behave, in the long term, and for any input

We can provide a general characterization of the long-term behavior of the output, as increasing or decreasing, with constant or alternating sign, for any finite input to the system

We will begin by studying the *unit-sample* response of systems, and then generalize to more general inputs

First-Order Systems:

Systems that only have forward connection can only have a finite response

The output signal will only have a finite number on non-zero samples

Systems with feedback have surprisingly different characters

Finite inputs can result in *persistent response*, meaning they can result in output signals with infinitely many non-zero samples

The qualitative long-term behavior of this output is generally independent of the particular input given to the system, for any finite input

We will consider the class of *first-order* systems, in which the denominator of the system function is a first-order polynomial

First order polynomial means it only involves R, but not R^2 or other higher powers of R

Example: **Y = X + pRY —> Y(1 - pR) = X —> Y = X÷(1 - pR)**

Deriving a system functional we get **H = Y/X = 1/(1 - pR)**

System responses can be characterized by a single number, called the *pole*, which is the base of the geometric sequence

The value of the pole, p, determines the nature and rate of growth

- If $p < -1$, the magnitude increases to infinite and the sign alternates
- If $-1 < p < 0$, the magnitude decreases and the sign alternates
- If $0 < p < 1$, the magnitude decreases monotonically
- If $p > 0$, the magnitude increases monotonically to infinity

Any system for which the output increases to infinity, whether monotonically or not, is called *unstable*

Systems whose output magnitude decreases or stays constant are called *stable*

Second-Order Systems

We will call these persistent long-term behaviors of a signal *modes*

For a fixed p, the first-order system only exhibited one mode, but different values of p resulted in very different modes

Second-order systems are characterized by a system functional whose denominator polynomial is second order

They will generally exhibit two modes

We will find that it is the mode whose pole has the largest magnitude that governs the long-term behavior of the system

We will call this pole the *dominant pole*

Additive Decomposition

Another way to try to decompose the system is as the *sum* of two simpler systems

In this case, we seek **H1** and **H2** such that **H1 + H2 = H**

Complex Poles

Difference equations that represent physical systems have real-valued coefficents, but they might still have complex poles

Polar Representation of Complex Numbers

Sometimes it is easier to think about a complex number as **a + bj**, where **a = rcosΩ** and **b = rsinΩ**, so that the *magnitude*, r, sometimes written as

$|a + bj|$, is defined as $r = \sqrt{a^2 + b^2}$ and that the angle, Ω, is defined as $\Omega = \tan^{-1}(b, a)$