hence $\sigma(\mathcal{E}) = \sigma(\mathcal{H}^+)$. For a fixed h and C, the continuous function $(1 - (h/C)^n)^+$ of h belongs to $\mathcal{H}^+$, and it increases monotonely to the indicator of $\{h < C\}$. Thus the indicators of all sets in $\mathcal{E}$ belong to $\mathcal{H}^+$. The assumptions about $\mathcal{H}^+$ ensure that the class $\mathcal{B}$ of all sets whose indicator functions belong to $\mathcal{H}^+$ is stable under finite intersections (products), complements (subtract from 1), and increasing countable unions (montone increasing limits). That is, $\mathcal{B}$ is a λ-system, stable under finite intersections, and containing $\mathcal{E}$. It is a sigma-field containing $\mathcal{E}$. Thus $\mathcal{B} \supseteq \sigma(\mathcal{E}) = \sigma(\mathcal{H}^+)$. That is, $\mathcal{H}^+$ contains all indicators of sets in $\sigma(\mathcal{H}^+)$.

Finally, let k be a bounded, nonnegative, $\sigma(\mathcal{H}^+)$-measurable function. From the fact that each of the sets $\{k \geq i/2^n\}$, for $i = 1, \ldots, 4^n$, belongs to the cone $\mathcal{H}^+$, we have $k_n := 2^{-n} \sum_{i=1}^{4^n} \{k \geq i/2^n\} \in \mathcal{H}^+$. The functions k_n increase monotonely to k, which consequently also belongs to $\mathcal{H}^+$. □

<45> **Theorem.** *Let $\mathcal{H}^+$ be a λ-cone of bounded, nonnegative functions, and $\mathcal{G}$ be a subclass of $\mathcal{H}^+$ that is stable under the formation of pointwise products of pairs of functions. Then $\mathcal{H}^+$ contains all bounded, nonnegative, $\sigma(\mathcal{G})$-measurable functions.*

Proof. Let $\mathcal{H}_0^+$ be the smallest λ-cone containing $\mathcal{G}$. From the previous Lemma, it is enough to show that $\mathcal{H}_0^+$ is stable under pairwise products.

Argue as in Theorem <38> for λ-systems of sets. A routine calculation shows that $\mathcal{H}_1^+ := \{h \in \mathcal{H}_0^+ : hg \in \mathcal{H}_0^+ \text{ for all } g \text{ in } \mathcal{G}\}$ is a λ-cone containing $\mathcal{G}$, and hence $\mathcal{H}_1^+ = \mathcal{H}_0^+$. That is, $h_0 g \in \mathcal{H}_0^+$ for all $h_0 \in \mathcal{H}_0^+$ and $g \in \mathcal{G}$. Similarly, the class $\mathcal{H}_2^+ := \{h \in \mathcal{H}_0^+ : h_0 h \in \mathcal{H}_0^+ \text{ for all } h_0 \text{ in } \mathcal{H}_0^+\}$ is a λ-cone. By the result for $\mathcal{H}_1^+$ we have $\mathcal{H}_2^+ \supseteq \mathcal{G}$, and hence $\mathcal{H}_2^+ = \mathcal{H}_0^+$. That is, $\mathcal{H}_0^+$ is stable under products. □

<46> **Exercise.** Let μ be a finite measure on $\mathcal{B}(\mathbb{R}^k)$. Write $\mathbf{C}_0$ for the vector space of all continuous real functions on $\mathbb{R}^k$ with compact support. Suppose f belongs to $\mathcal{L}^1(\mu)$. Show that for each $\epsilon > 0$ there exists a g in $\mathbf{C}_0$ such that $\mu|f - g| < \epsilon$. That is, show that $\mathbf{C}_0$ is dense in $\mathcal{L}^1(\mu)$ under its $\mathcal{L}^1$ norm. □

SOLUTION: Define $\mathcal{H}$ as the collection of all bounded functions in $\mathcal{L}^1(\mu)$ that can be approximated arbitrarily closely by functions from $\mathbf{C}_0$. Check that the class $\mathcal{H}^+$ of nonnegative functions in $\mathcal{H}$ is a λ-cone. Trivially it contains $\mathbf{C}_0^+$, the class of nonnegative members of $\mathbf{C}_0$. The sigma-field $\sigma(\mathbf{C}_0^+)$ coincides with the Borel sigma-field. Why? The class $\mathcal{H}^+$ consists of all bounded, nonnegative Borel measurable functions.

To approximate a general f in $\mathcal{L}^1(\mu)$, first reduce to the case of nonnegative functions by splitting into positive and negative parts. Then invoke Dominated Convergence to find a finite n for which $\mu|f^+ - f^+ \wedge n| < \epsilon$, then approximate $f^+ \wedge n$ by a member of $\mathbf{C}_0^+$. See Problem [26] for the extension of the approximation result to infinite measures. □

12. Problems

[1] Suppose events $A_1, A_2, \ldots$, in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, are independent: meaning that $\mathbb{P}(A_{i_1} A_{i_2} \ldots A_{i_k}) = \mathbb{P}A_{i_1} \mathbb{P}A_{i_2} \ldots \mathbb{P}A_{i_k}$ for all choices of distinct subscripts $i_1, i_2, \ldots, i_k$, all k. Suppose $\sum_{i=1}^{\infty} \mathbb{P}A_i = \infty$.

(i) Using the inequality $e^{-x} \geq 1 - x$, show that

$$\mathbb{P} \max_{n \leq i \leq m} A_i = 1 - \prod_{n \leq i \leq m} (1 - \mathbb{P}A_i) \geq 1 - \exp\left(- \sum_{n \leq i \leq m} \mathbb{P}A_i \right)$$

(ii) Let m then n tend to infinity, to deduce (via Dominated Convergence) that $\mathbb{P} \limsup_i A_i = 1$. That is, $\mathbb{P}\{A_i \text{ i. o.}\} = 1$.

> REMARK. The result gives a converse for the Borel-Cantelli lemma from Example <29>. The next Problem establishes a similar result under weaker assumptions.

[2] Let $A_1, A_2, \ldots$ be events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $X_n = A_1 + \ldots + A_n$ and $\sigma_n = \mathbb{P}X_n$. Suppose $\sigma_n \to \infty$ and $\|X_n/\sigma_n\|_2 \to 1$. *(Compare with the inequality $\|X_n/\sigma_n\|_2 \geq 1$, which follows from Jensen's inequality.)*

(i) Show that

$$\{X_n = 0\} \leq \frac{(k - X_n)(k + 1 - X_n)}{k(k+1)}$$

for each positive integer k.

(ii) By an appropriate choice of k (depending on n) in (i), deduce that $\sum_1^\infty A_i \geq 1$ almost surely.

(iii) Prove that $\sum_m^\infty A_i \geq 1$ almost surely, for each fixed m. Hint: Show that the two convergence assumptions also hold for the sequence $A_m, A_{m+1}, \ldots$.

(iv) Deduce that $\mathbb{P}\{\omega \in A_i \text{ i. o. }\} = 1$.

(v) If $\{B_i\}$ is a sequence of events for which $\sum_i \mathbb{P}B_i = \infty$ and $\mathbb{P}B_iB_j = \mathbb{P}B_i\mathbb{P}B_j$ for $i \neq j$, show that $\mathbb{P}\{\omega \in B_i \text{ i. o. }\} = 1$.

[3] Suppose T is a function from a set $\mathcal{X}$ into a set $\mathcal{Y}$, and suppose that $\mathcal{Y}$ is equipped with a σ-field $\mathcal{B}$. Define $\mathcal{A}$ as the sigma-field of sets of the form $T^{-1}B$, with B in $\mathcal{B}$. Suppose $f \in \mathcal{M}^+(\mathcal{X}, \mathcal{A})$. Show that there exists a $\mathcal{B}\backslash\mathcal{B}[0, \infty]$-measurable function g from $\mathcal{Y}$ into $[0, \infty]$ such that $f(x) = g(T(x))$, for all x in $\mathcal{X}$, by following these steps.

(i) Show that $\mathcal{A}$ is a σ-field on $\mathcal{X}$. (It is called the σ-field generated by the map T. It is often denoted by $\sigma(T)$.)

(ii) Show that $\{f \geq i/2^n\} = T^{-1}B_{i,n}$ for some $B_{i,n}$ in $\mathcal{B}$. Define

$$f_n = 2^{-n} \sum_{i=1}^{4^n} \{f \geq i/2^n\} \qquad \text{and} \qquad g_n = 2^{-n} \sum_{i=1}^{4^n} B_{i,n}.$$

Show that $f_n(x) = g_n(T(x))$ for all x.

(iii) Define $g(y) = \limsup g_n(y)$ for each y in $\mathcal{Y}$. Show that g has the desired property. (Question: Why can't we define $g(y) = \lim g_n(y)$?)

[4] Let $g_1, g_2, \ldots$ be $\mathcal{A}\backslash\mathcal{B}(\mathbb{R})$-measurable functions from $\mathcal{X}$ into $\mathbb{R}$. Show that $\{\limsup_n g_n > t\} = \bigcup_{\substack{r \in \mathbb{Q} \\ r > t}} \bigcap_{m=1}^\infty \bigcup_{i \geq m} \{g_i > r\}$. Deduce, without any appeal to Example <8>, that $\limsup g_n$ is $\mathcal{A}\backslash\mathcal{B}(\overline{\mathbb{R}})$-measurable. Warning: Be careful about

strict inequalities that turn into nonstrict inequalities in the limit—it is possible to have $x_n > x$ for all n and still have $\limsup_n x_n = x$.

[5] Suppose a class of sets $\mathcal{E}$ cannot separate a particular pair of points x, y: for every E in $\mathcal{E}$, either $\{x, y\} \subseteq E$ or $\{x, y\} \subseteq E^c$. Show that $\sigma(\mathcal{E})$ also cannot separate the pair.

[6] A collection of sets $\mathcal{F}_0$ that is stable under finite unions, finite intersections, and complements is called a field. A nonnegative set function μ defined on $\mathcal{F}_0$ is called a finitely additive measure if $\mu\left(\cup_{i\le n} F_i\right) = \sum_{i\le n} \mu F_i$ for every finite collection of disjoint sets in $\mathcal{F}_0$. The set function is said to be countably additive on $\mathcal{F}_0$ if $\mu\left(\cup_{i\in\mathbb{N}} F_i\right) = \sum_{i\in\mathbb{N}} \mu F_i$ for every countable collection of disjoint sets in $\mathcal{F}_0$ *whose union belongs to* $\mathcal{F}$. Suppose $\mu\mathcal{X} < \infty$. Show that μ is countably additive on $\mathcal{F}_0$ if and only if $\mu A_n \downarrow 0$ for every decreasing sequence in $\mathcal{F}_0$ with empty intersection. Hint: For the argument in one direction, consider the union of differences $A_i \backslash A_{i+1}$.

[7] Let $f_1, \ldots, f_n$ be functions in $\mathcal{M}^+(\mathcal{X}, \mathcal{A})$, and let μ be a measure on $\mathcal{A}$. Show that $\mu\left(\vee_i f_i\right) \le \sum_i \mu f_i \le \mu\left(\vee_i f_i\right) + \sum_{i<j} \mu\left(f_i \wedge f_j\right)$ where $\vee$ denotes pointwise maxima of functions and $\wedge$ denotes pointwise minima.

[8] Let μ be a finite measure and f be a measurable function. For each positive integer k, show that $\mu|f|^k < \infty$ if and only if $\sum_{n=1}^{\infty} n^{k-1} \mu\{|f| \ge n\} < \infty$.

[9] Suppose $\nu := T\mu$, the image of the measure μ under the measurable map T. Show that $f \in \mathcal{L}^1(\nu)$ if and only if $f \circ T \in \mathcal{L}^1(\mu)$, in which case $\nu f = \mu\left(f \circ T\right)$.

[10] Let $\{h_n\}$, $\{f_n\}$, and $\{g_n\}$ be sequences of μ-integrable functions that converge μ almost everywhere to limits h, f and g. Suppose $h_n(x) \le f_n(x) \le g_n(x)$ for all x. Suppose also that $\mu h_n \to \mu h$ and $\mu g_n \to \mu g$. Adapt the proof of Dominated Convergence to prove that $\mu f_n \to \mu f$.

[11] A collection of sets is called a monotone class if it is stable under unions of increasing sequences and intersections of decreasing sequences. Adapt the argument from Theorem <38> to prove: if a class $\mathcal{E}$ is stable under finite unions and complements then $\sigma(\mathcal{E})$ equals the smallest monotone class containing $\mathcal{E}$.

[12] Let μ be a finite measure on the Borel sigma-field $\mathcal{B}(\mathcal{X})$ of a metric space $\mathcal{X}$. Call a set B ***inner regular*** if $\mu B = \sup\{\mu F : B \supseteq F \text{ closed }\}$ and ***outer regular*** if $\mu B = \inf\{\mu F : B \subseteq G \text{ open }\}$

(i) Prove that the class $\mathcal{B}_0$ of all Borel sets that are both inner and outer regular is a sigma-field. Deduce that every Borel set is inner regular.

(ii) Suppose μ is tight: for each $\epsilon > 0$ there exists a compact K_ϵ such that $\mu K_\epsilon^c < \epsilon$. Show that the F in the definition of inner regularity can then be assumed compact.

(iii) When μ is tight, show that there exists a sequence of disjoint compacts subsets $\{K_i : i \in \mathbb{N}\}$ of $\mathcal{X}$ such that $\mu\left(\cup_i K_i\right)^c = 0$.

[13] Let μ be a finite measure on the Borel sigma-field of a complete, separable metric space $\mathcal{X}$. Show that μ is tight: for each $\epsilon > 0$ there exists a compact K_ϵ such that $\mu K_\epsilon^c < \epsilon$. Hint: For each positive integer n, show that the space $\mathcal{X}$ is a countable