Functions

$f : A \to B$: a function f from A to B is an assignment of exactly one element of B to each element of A. Write $f(a) = b$.

e.g. Real functions, birthday, height in centimeters, SSN.

A is the *domain*, B is the *codomain*, a is a preimage of b, and b is the image of a. Range of f is the set $\{b \in B : \exists a \in A(f(a) = b)\}$.

Two functions are equal if they have the same domain, codomain, and assignment rule.

Example 1. *1. $f(x) = x^2$ on $\mathbb{Z}$, or on $\mathbb{R}$.*

2. $f(x) = \sqrt{x^2}$, $g(x) = |x|$, and $h(x) = (\sqrt{x})^2$. Then f=g, but not h.

Sometimes the domain/codomain are not specified. Then just take whatever they are defined. e.g. $f(x) = \sqrt{1 - x^2}$, or $g(x) = 1/[(1 + x)(2 - x)]$.

Image and preimage Let $f : A \to B$. For a subset C of A, we define the image of C as $f(C) = \{b \in B : \exists c \in C(f(c) = b)\}$.

For a subset D of B, we define the preimage of D as $f^{-1}(D) = \{a \in A : f(a) \in D\}$.

Note that they are sets, always well-defined. e.g. $f(x) = x^2$. The images of $(1, 2)$ is $(1, 4)$, the preimage of $(1, 4)$ is $(-2, -1) \cup (1, 2)$.

We have

$$f^{-1}(f(C)) \supseteq C, \qquad F(f^{-1}(D)) \subseteq D$$

The equation may not hold.

One-to-one functions

A function is one-to-one, or *injective*, if and only if $f(a) = f(b)$ implies $a = b$ for all a, b in the domain.

Using contrapositive, f is 1-1 iff $f(a) \neq f(b)$ whenever $a \neq b$. But the definition is easier to check.

e.g. $f(x) = x^2$, $f(x) = 3x + 7$, and $f(x) = x/(x + 1)$.

Remark. (i) Whether 1-1 depends on the domain/codomain. (ii) If A and B are finite and $f : A \to B$ is 1-1, then $|A| \leq |B|$.

Onto functions

A function is called *onto*, or *surjective*, if and only if its range equals its codomain. Or $\forall b \in B$, $f^{-1}(\{b\}) \neq \emptyset$.

e.g. $f(x) = x^3$, $f(x) = \tan x$, $f(x) = 3x + 7$. (To check, solve x from y.)

Remark (i) Whether onto depends on the domain/codomain. (ii) If A and B are finite and $f : A \to B$ is onto, then $|A| \geq |B|$.

A function f is a one-to-one correspondence, or *bijection*, if it is both one-to-one and onto. In that case, for two finite sets, $|A| = |B|$.

Functions with $\cap$ and $\cup$

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2), \qquad f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

How to prove the first one?

$$
\begin{aligned}
& b \in f(A_1 \cup A_2) \\
\Leftrightarrow \quad & \exists a(f(a) = b \wedge (a \in A_1 \cup A_2)) \\
\Leftrightarrow \quad & \exists a(f(a) = b \wedge (a \in A_1 \vee a \in A_2)) \\
\Leftrightarrow \quad & \exists a[(f(a) = b \wedge a \in A_1) \vee (f(a) = b \wedge a \in A_2)] \\
\Leftrightarrow \quad & b \in f(A_1) \vee b \in f(A_2) \\
\Leftrightarrow \quad & b \in f(A_1) \cup f(A_2)
\end{aligned}
$$

It is easier to use plain English. e.g.

" $b \in f(A_1 \cup A_2)$ implies that there is a preimage a of b s.t. $a \in A_1 \cup A_2$. Then $a \in A_1$ or $a \in A_2$, which means $b \in f(A_1)$ or $b \in f(A_2)$. So b is in $f(A_1) \cup f(A_2)$. "

For the second one, e.g., $f(x) = x^2$, and $A_1 = \mathbf{R}_+$, $A_2 = \mathbf{R}_-$.

$$
f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2), \qquad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)
$$

It is in HW5. Hint: $a \in f^{-1}(D)$ if and only if $f(a) \in D$.

For the first one, a proof:
" for all a, $a \in f^{-1}(B_1 \cup B_2)$ iff $f(a) \in B_1 \cup B_2$, which is, $f(a) \in B_1$ or $f(a) \in B_2$. It is equivalent to say that $a \in f^{-1}(B_1)$ or $a \in f^{-1}(B_2)$, which is exactly $a \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

Special Functions.

1. The identity map: $f : A \to A$, $f(x) = x$. Write it as I_A.

2. Floor and ceiling functions
$\lfloor x \rfloor =$ the largest integer y such that $y \leq x$.
$\lceil x \rceil =$ the smallest integer y such that $y \geq x$.

e.g. for $x = 10, 100, \pi, -2, -2.5$.

Property.

1. $\lfloor -x \rfloor = -\lceil x \rceil$, $\lceil -x \rceil = -\lfloor x \rfloor$
2. $x - 1 \leq \lfloor x \rfloor \leq x \leq \lceil x \rceil \leq x + 1$
3. $\lceil x + n \rceil = \lceil x \rceil + n$, $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for integers n. (not true for an arbitrary real number n.