The basis in a nutshell 2/28/2001 **Math 21b, O. Knill**

DEFINITION BASIS. A set of vectors $v_1, \dots, v_n$ is called a **basis** of a linear space if they are **linear independent** and if they **span** the linear space. Linear independent means that there are no **linear relations** $a_i v_1 + \dots + a_n v_n = 0$. Spanning the space means that very vector v can be written as a linear combination $v = a_1 v_1 + \dots + a_n v_n$ of the basis.

FACT. If $v_1, ..., v_n$ is a basis, then every vector v can be represented **uniquely** as a linear combination of the v_i. $v = a_1 v_1 + \dots + a_n v_n$.

REASON. The representation is possible because the v_i span the space. If there were two different representations, then subtracting these equation would lead to a linear relation between the v_i which is forbidden by the linear independence.

FACT. If n vectors $v_1, ..., v_n$ span a space and $w_1, ..., w_m$ are linear independent vectors, then $m \leq n$. (see p. 162 top in the book)

DIMENSION. The number of elements in a basis is independent of the basis. It is called the **dimension** of the linear space.

BASIS DEFINES INVERTIBLE MATRIX. The $n \times n$ matrix $A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}$ is invertible if and only if $v_1, \dots, v_n$ define a basis in $\mathbf{R}^n$.

DIMENSION OF THE KERNEL. The number of **non-leading entries** rref(A) is the **dimension of the kernel** $\dim(\ker(A))$ because we can introduce a parameter for each of those when solving $Ax = 0$ using Gauss elimination.

DIMENSION OF THE IMAGE. The number of **leading 1** in rref(A), the rank of A is the **dimension of the image** $\dim(\mathrm{im}(A))$ because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.

DIMENSION FORMULA:

$$\dim(\mathbf{ker}(A)) + \dim(\mathrm{im}(A)) = n$$

if A is a $m \times n$ matrix. The number n is the dimension of the domain $\mathbf{R}^n$ of $A : \mathbf{R}^n \to \mathbf{R}^m$.

EXAMPLE: A matrix A is invertible if and only if the dimension of the image is n. Therefore, the dimension of the kernel is 0.(We know that also because we have a unique solution to $Ax = 0$).

FINDING A BASIS FOR THE KERNEL. To solve $Ax = 0$, we bring the matrix A into the reduced row echelon form rref(A). For every non-leading entry in rref(A), we will get a free variable t_i. Writing the system $Ax = 0$ with these free variables gives us an equation $\vec{x} = \sum_i t_i \vec{v_i}$. The vectors v_i form a basis of the kernel of A.
REMARK. If we have the problem to find a basis for all vectors w_i which are orthogonal to a given set of vectors, this is the problem to find a basis for the kernel of the matrix which has the vectors w_i in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the $m \times n$ matrix A into the form rref(A). Call a column a **pivot column**, if it contains a leading 1. The corresponding set of column vectors of the original matrix A form a basis for the image because they are linearly independent and are in the image. Assume there are k of them. They span the image because there are $(k - n)$ non-leading entries in the matrix.
REMARK. If we have the problem to find a basis of the subspace generated by a few vectors $v_1, \dots, v_n$, then this is the problem to find a basis for the matrix with column vectors $v_1, ... v_n$.

Homework for Friday, March 2: Section: 3.3, Numbers 22,24,32,36,40,56*

Examples

EXAMPLE. The vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis in the three dimensional space.

EXAMPLE. If $v = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, then $v = v_1 + 2v_2 + 3v_3$ (with the basis above) and this representation is unique. We can find the coefficients by solving $Ax = v$, where A has the v_i as column vectors. In our case, $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ had the unique solution $x = 1, y = 2, z = 3$ leading to $v = v_1 + 2v_2 + 3v_3$. The matrix A is invertible because v_1, v_2, v_3 form a basis.

EXAMPLE. Two vectors on a line are linear dependent. The linear relation says that one is a multiple of the other. Three vectors in the plane are linear dependent. One can find a relation $av_1 + bv_2 = v_3$ by changing the size of the lengths of the vectos v_1, v_2 until v_3 becomes the diagonal of the parallelogram spanned by v_1, v_2. Four vectors in three dimensional space are linearly dependent. As in the plane one can change the length of the vectors to make v_4 a diagonal of the parallelepiped spanned by v_1, v_2, v_3.

EXAMPLES. The dimension of the point $\{0\}$ is zero. The dimension of a line is 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. A line in the plane and a line in space for example have the same dimension 1.

EXAMPLE. Let A be the matrix $A = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$. In reduced row echelon form it is $\text{rref}(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$. There is one **non-leading entry** $A_{12} = 1$ in the first row. The dimension of the kernel is 1.

EXAMPLE. In the example, there are two pivotal column vectors, the first and the third. Therefore, the dimension of the image is 2. The dimension formula $2 + 1 = 3$ is satisfied.

EXAMPLE. $B = \text{rref}(A) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$. We determine a basis of the kernel by writing the $Bx = 0$ as a system of linear equations $x + y = 0, z = 0$. The variable y is a free variable. With $y = t$, we have $x = -t$. The linear system $\text{rref}(A)x = 0$ is solved by $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel.

EXAMPLE. In order to get a basis of the image, we chose the first and third column vectors of the original matrix A because the first and third vectors in $\text{rref}(A)$ are pivot columns. So, $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of the image of A.