

The Complex Propagation Constant γ

Recall that the current and voltage along a transmission line have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

where Z_0 and γ are **complex constants** that describe the properties of a transmission line. Since γ is complex, we can consider both its **real** and **imaginary** components.

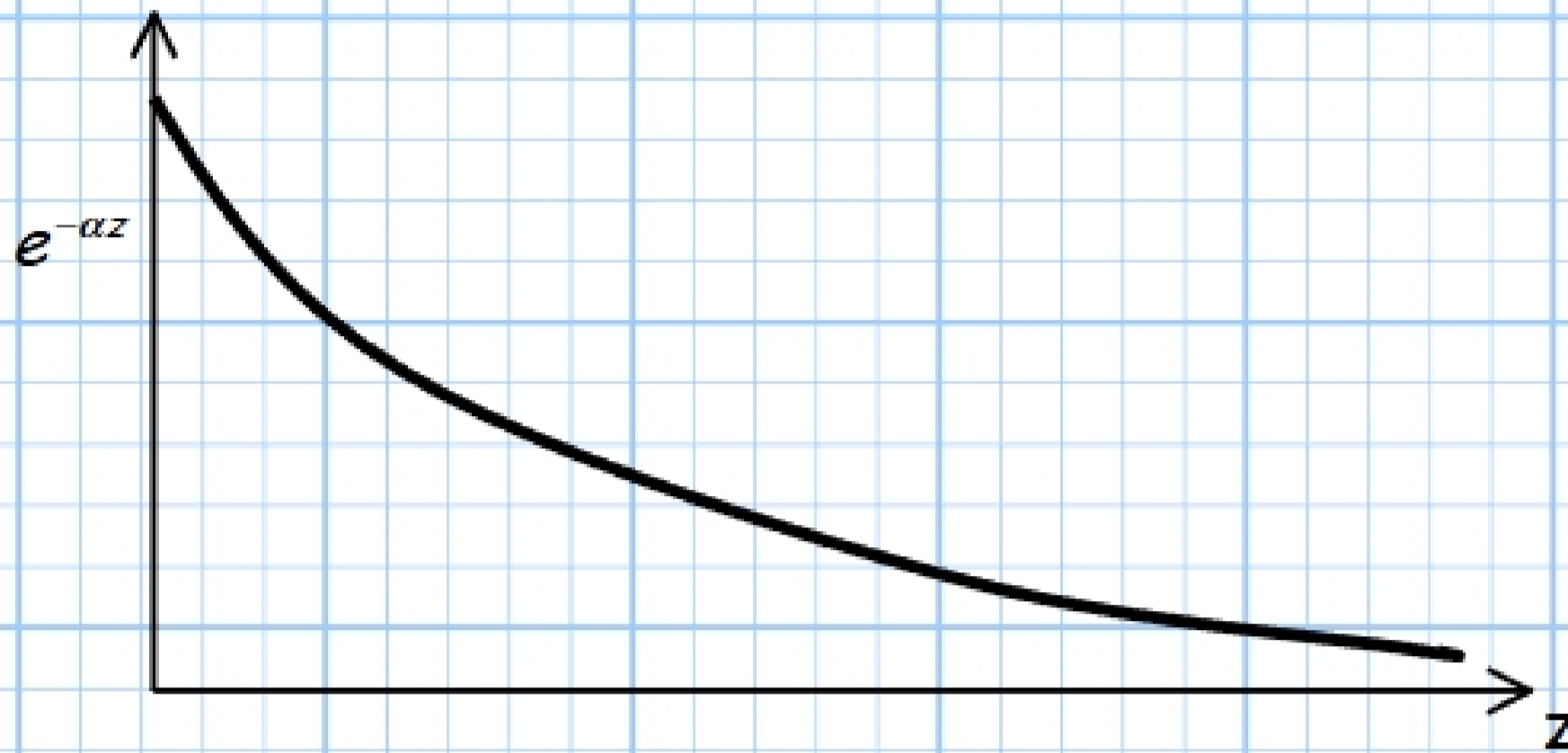
$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \alpha + j\beta \end{aligned}$$

where $\alpha = \text{Re}\{\gamma\}$ and $\beta = \text{Im}\{\gamma\}$. Therefore, we can write:

$$e^{-\gamma z} = e^{-(\alpha + j\beta)z} = e^{-\alpha z} e^{-j\beta z}$$

Since $|e^{-j\beta z}| = 1$, then $e^{-\alpha z}$ alone determines the magnitude of $e^{-\gamma z}$.

$$\text{I.E., } |e^{-\gamma z}| = e^{-\alpha z}.$$



Therefore, α expresses the **attenuation** of the signal due to the loss in the transmission line.

Since $e^{-\alpha z}$ is a real function, it expresses the **magnitude** of $e^{-\gamma z}$ only. The **relative phase** $\phi(z)$ of $e^{-\gamma z}$ is therefore determined by $e^{-j\beta z} = e^{-j\phi(z)}$ only (recall $|e^{-j\beta z}| = 1$).

From Euler's equation:

$$e^{j\phi(z)} = e^{j\beta z} = \cos(\beta z) + j \sin(\beta z)$$

Therefore, βz represents the **relative phase** $\phi(z)$ of the oscillating signal, as a function of transmission line position z . Since phase $\phi(z)$ is expressed in radians, and z is distance (in meters), the value β must have units of :

$$\beta = \frac{\phi}{z} \quad \frac{\text{radians}}{\text{meter}}$$

The **wavelength** λ of the signal is the distance $\Delta z_{2\pi}$ over which the relative phase changes by 2π radians. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta \Delta z_{2\pi} = \beta \lambda$$

or, rearranging:

$$\beta = \frac{2\pi}{\lambda}$$

Since the signal is oscillating in **time** at rate ω rad/sec, the **propagation velocity** of the wave is:

$$v_p = \frac{\omega}{\beta} = \frac{\omega \lambda}{2\pi} = f \lambda \quad \left(\frac{\text{m}}{\text{sec}} = \frac{\text{rad}}{\text{sec}} \frac{\text{m}}{\text{rad}} \right)$$

where f is **frequency** in cycles/sec.

Recall we originally considered the transmission line current and voltage as a function of time and position (i.e., $v(z, t)$ and $i(z, t)$). We assumed the time function was sinusoidal, oscillating with frequency ω :

$$v(z, t) = \text{Re} \{ V(z) e^{j\omega t} \}$$

$$i(z, t) = \text{Re} \{ I(z) e^{j\omega t} \}$$