2/7/2002, MATRICES AND GAUSS-JORDAN — Math 21b, O. Knill

HOMEWORK: 1.2: 6,11,18,20,30,38*, 1.3:4,14,26*,34,48,50, **Due:** Tue 2/12/2002

MATRIX REFORMULATION. Consider the system of linear equations

$$\left| \begin{array}{rl} 3x - y - z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right|$$

The system can be written as $A\vec{x} = \vec{b}$, where A is a **matrix** (called **coefficient matrix**) and and $\vec{x}$ and $\vec{b}$ are **vectors**.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

We also look at the **augmented matrix** (the seperators are there for clarity reasons only)

$$B = \left[\begin{array}{rrr|r} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

MATRIX "JARGON". A rectangular array of numbers is called a **matrix**. If the matrix has m **rows** and n **columns**, it is called a $m \times n$ matrix. A matrix with one column only is called a **column vector**. The entries of a matrix are denoted by a_{ij}, where i is the row and j is the column. In the case of the linear equation above, the matrix A is a square matrix and the augmented matrix B above is a 3×4 matrix.

REWRITING A LINEAR EQUATION.

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \cdots \\ -\vec{w}_m- \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \cdots \\ \vec{w}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}.$$

In words: the entries b_i are the dot product of row vectors $\vec{w}_i$ with $\vec{x}$. The vector $\vec{b}$ is a sum of scaled column vectors $\vec{v}_j$ of A.

GAUSS-JORDAN ELIMINATION. Gauss-Jordan Elimination is a process, where successive subtraction of multiples of other rows or scaling brings the matrix into a so-called **reduced row echelon form**. The elimination process consists of three possible steps which are called **elementary row operations**:

- Swap two rows.
- Divide a row by a scalar
- Subtract a multiple of a row from an other row.

The process transfers a given matrix A into a new matrix rref(A) in reduced row echelon form.

REDUCED ECHELON FORM. A matrix is called in **reduced row echelon form**

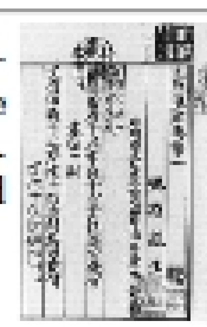
1) if a row has nonzero entries, then the first nonzero entry is 1. (**leading one**)
2) if a column contains a leading 1, then the other column entries are 0.
3) if a row has a leading 1, then every row above has leading 1's to the left.

RANK. The number of leading 1 in rref(A) is called the rank of A.

SOLUTIONS OF LINEAR EQUATIONS. If $Ax = b$ is a linear system of equations with m equations and n unknowns, then A is a $m \times n$ matrix. We have the following three possibilities:

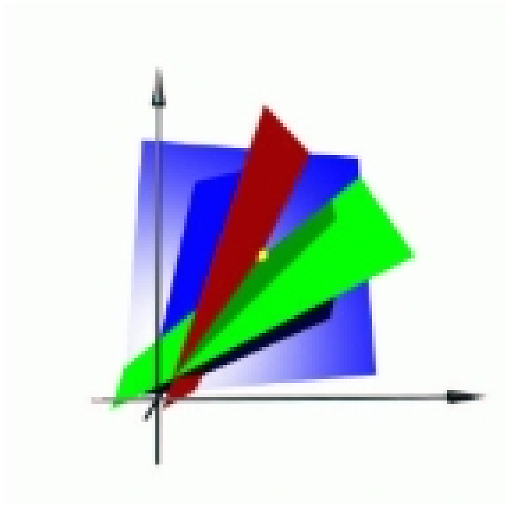
- **Exactly one solution.** There is a leading 1 in each row but not in the last row.
- **No solutions.** There is a leading 1 in the last row.
- **Infinitely many solutions.** There are rows without leading 1 and no leading 1 is in the last row.

JIUZHANG SUANSHU. The technique of successively eliminating variables from systems of linear equations is called **Gauss elimination** and appeared already in the Chinese manuscript "Jiuzhang Suanshu" ('Nine Chapters on the Mathematical art'). The manuscript appeared around 200 BC in the Han dynasty and was probably used as a textbook. For more history of Chinese Mathematics, see http://aleph0.clarku.edu/~djoyce/mathhist/china.html.



EXAMPLES. The reduced echelon form of the augmented matrix B determines on how many solutions the linear system $Ax = b$ has.

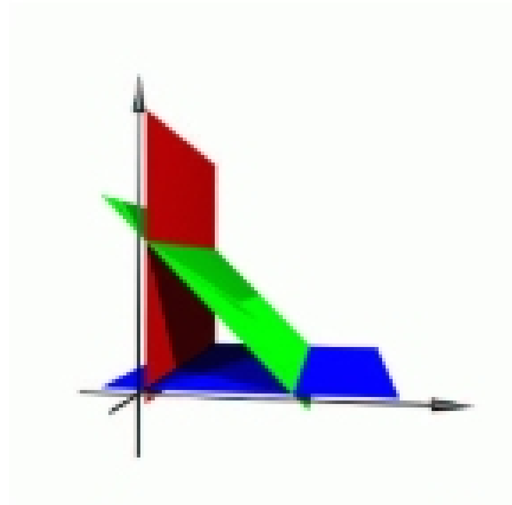
THE GOOD (exactly one solution)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & -1 & -2 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & -3 & -12 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -4 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\text{Rank}(A) = 3$, $\text{Rank}(B) = 3$.

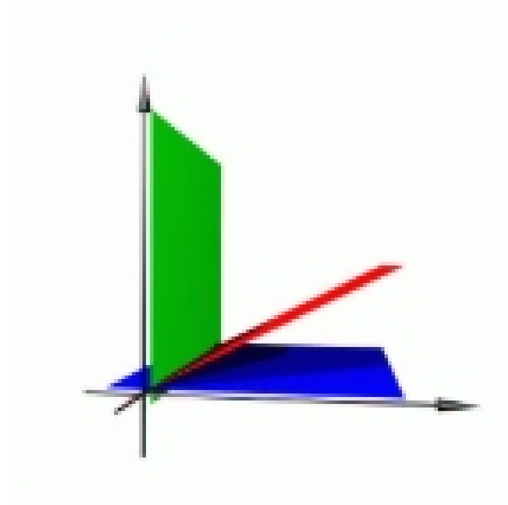
THE BAD (no solution)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & -2 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & -2 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -7 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\text{Rank}(A) = 2$, $\text{Rank}(B) = 3$.

THE UGLY (infinitely many solutions)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & 7 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 7 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{Rank}(A) = 2$, $\text{Rank}(B) = 2$.

JORDAN. The German geodesist Wilhelm Jordan (1842-1899) applied the Gauss-Jordan method to finding squared errors to work on surveying. (An other "Jordan", the French Mathematician Camille Jordan (1838-1922) worked on linear algebra topics also and is often mistakenly credited with the Gauss-Jordan process.)

GAUSS. Gauss developed Gaussian elimination around 1800 and used it to solve least squares problems in celestial mechanics and later in geodesic computations. In 1809, Gauss published the book "Theory of Motion of the Heavenly Bodies" in which he used the method for solving astronomical problems. One of Gauss successes was the prediction of an asteroid orbit.

On the 1. January of 1801, the Italian astronomer Giuseppe Piazzi (1746-1826) discovered Ceres, the first and largest known asteroid in our solar system. Ceres is a rock of 914 km diameter. (The picture shows a photo in infrared). Gauss was able to predict the orbit of Ceres from a few observations. By parameterizing the orbit with parameters and solving a linear system of equations (similar to one of the homework problems, where you will fit a cubic curve from 4 observations), he was able to derive the parameters.