

Gauss-Jordan

Given the following system of equations:

$$2x + 4y + 6z = 22$$

$$3x + 8y + 5z = 27$$

$$x + y + 2z = 2$$

the coefficient matrix is: $\begin{pmatrix} 2 & 4 & 6 \\ 3 & 8 & 5 \\ 1 & 1 & 2 \end{pmatrix}$

the augmented matrix is: $\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ 1 & 1 & 2 & 2 \end{array} \right]$ We will use the augmented matrix.

When using the Gauss-Jordan elimination method using very strict rules will look like:

$$\begin{array}{ccc|c} x & y & z & \\ \hline 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}$$

Where a , b and c (a , b and c are numbers) are the values for x , y and z respectively.

Row Operations

1. Interchange any two rows.

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 5 \end{bmatrix} R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \end{bmatrix}$$

2. Replace any row by a nonzero constant multiple of itself.

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -2 & 8 \end{bmatrix} \frac{-1}{2}R_2 \rightarrow R_2 \quad \begin{bmatrix} 2 & -1 & 3 \\ -2 & 1 & -4 \end{bmatrix}$$

3. Replace any row by the sum of that row and a constant multiple of any other row.

$$\begin{array}{l} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 3 \end{array} \right] \quad -2R_1 + R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -7 \end{array} \right]$$

Pivoting a Matrix About an Element

The sequence of row operations that transforms the augmented matrix into the equivalent matrix in which the 1st column is transformed into the unit column is called pivoting the matrix about the element that transformed into 1 (leading 1).

Example: Given $\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}$

To change the column $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ into the unit column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we perform

certain row operations and we call this pivoting the matrix about the element 2.

$$\begin{array}{l} \left[\begin{array}{cc} 2 & 4 \\ -1 & 2 \end{array} \right] \quad 2 \text{ needs to be a } 1 \quad \frac{1}{2}R_1 \rightarrow R_1 \quad \left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array} \right] \\ \left[\begin{array}{cc} 1 & 2 \\ -1 & 2 \end{array} \right] \quad -1 \text{ needs to be a zero} \quad R_1 + R_2 \rightarrow R_2 \\ \left[\begin{array}{cc} 1 & 2 \\ 0 & 4 \end{array} \right] \end{array}$$

Example 1: Pivot the following matrix about the circled element.

$$\left(\begin{array}{cc|c} 3 & 15 & 6 \\ 8 & 18 & 5 \end{array} \right) \quad \frac{1}{3}R_1 \rightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & 5 & 2 \\ 8 & 18 & 5 \end{array} \right] \quad -8R_1 + R_2 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & -22 & -11 \end{array} \right] \quad -\frac{1}{22}R_2 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 1 & \frac{1}{2} \end{array} \right] \quad 5 \text{ to be zero} \quad -5R_2 + R_1 \rightarrow R_1$$

$$-\frac{5}{2} + 2 = -\frac{5}{2} + \frac{4}{2} = -\frac{1}{2} \quad \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{array} \right] \quad \begin{array}{l} x = -\frac{1}{2} \\ y = \frac{1}{2} \end{array}$$

The Gauss-Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

$$\begin{array}{c} x \quad y \quad z \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right] \end{array} \quad \begin{array}{l} x = a \\ y = b \\ z = c \end{array}$$

Example 2: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{array}{l} x + 2y = 1 \\ 2x + 3y = -1 \end{array} \quad \begin{array}{l} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & -1 \end{array} \right] \quad -2R_1 + R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -3 \end{array} \right] \quad (-1)R_2 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 3 \end{array} \right] \quad -2R_2 + R_1 \rightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \end{array} \right] \quad \begin{array}{l} x = -5 \\ y = 3 \end{array}$$

Example 3: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$\begin{array}{l} 3x + y = 1 \\ -7x - 2y = -1 \end{array} \quad \left[\begin{array}{cc|c} 3 & 1 & 1 \\ -7 & -2 & -1 \end{array} \right] \quad (\frac{1}{3})R_1 \rightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ -7 & -2 & -1 \end{array} \right] \quad \begin{array}{l} 7R_1 + \\ R_2 \rightarrow R_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{4}{3} \end{array} \right] \quad 3R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 4 \end{array} \right] \quad -\frac{1}{3}R_2 + R_1 \rightarrow R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} x = -\frac{1}{3} \\ y = 4 \end{array}$$

$$\begin{array}{l} 7 \cdot \frac{1}{3} - 2 = \frac{7}{3} - \frac{6}{3} = \frac{1}{3} \\ 7 \cdot \frac{1}{3} - 1 = \frac{7}{3} - \frac{3}{3} = \frac{4}{3} \end{array}$$