Lecture #8: Computational Methods for Plasma Physics

I. Simulation of Single Particle Motion

A. Particle Motion in Specified fields $\underset{\sim}{E}(\underset{\sim}{x},t)$ and $\underset{\sim}{B}(\underset{\sim}{x},t)$

1. Lorentz Force Law $\qquad \frac{d\underset{\sim}{v}(t)}{dt} = \frac{q}{m}\left[\underset{\sim}{E}(\underset{\sim}{x},t) + \underset{\sim}{v}(t)\times \underset{\sim}{B}(\underset{\sim}{x},t)\right]$

2. Velocity $\qquad \frac{d\underset{\sim}{x}(t)}{dt} = \underset{\sim}{v}(t)$

3. This is a set of <u>Ordinary differential equations</u> (ODEs) which can be solved, if $\underset{\sim}{E}(\underset{\sim}{x},t)$ & $\underset{\sim}{B}(\underset{\sim}{x},t)$ are known, to advance $\underset{\sim}{x}(t)$ and $\underset{\sim}{v}(t)$ in time.

4. This requires <u>initial conditions</u> at $t=0$, $\underset{\sim}{x}(t=0)$ and $\underset{\sim}{v}(t=0)$.

5. In general, any ODE problem can be reduced to a set of first-order equations:

a. Ex: $\qquad \frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$

$$\Rightarrow \begin{cases} \frac{dy}{dx} = z(x) & \longleftarrow \text{Define new variable } z(x). \\ \frac{dz}{dx} = r(x) - q(x)\,z(x) \end{cases}$$

b. The general form, then, is for a set of N coupled first-order differential equations for functions
$$y_i \quad , \quad i = 1,2,\ldots,N$$
where
$$\frac{dy_i(x)}{dx} = f_i(x, y_1, y_2, \ldots, y_N)$$
and the functions f_i are known.

c. <u>Boundary Values</u> on functions y_i are required to specify solution fully.

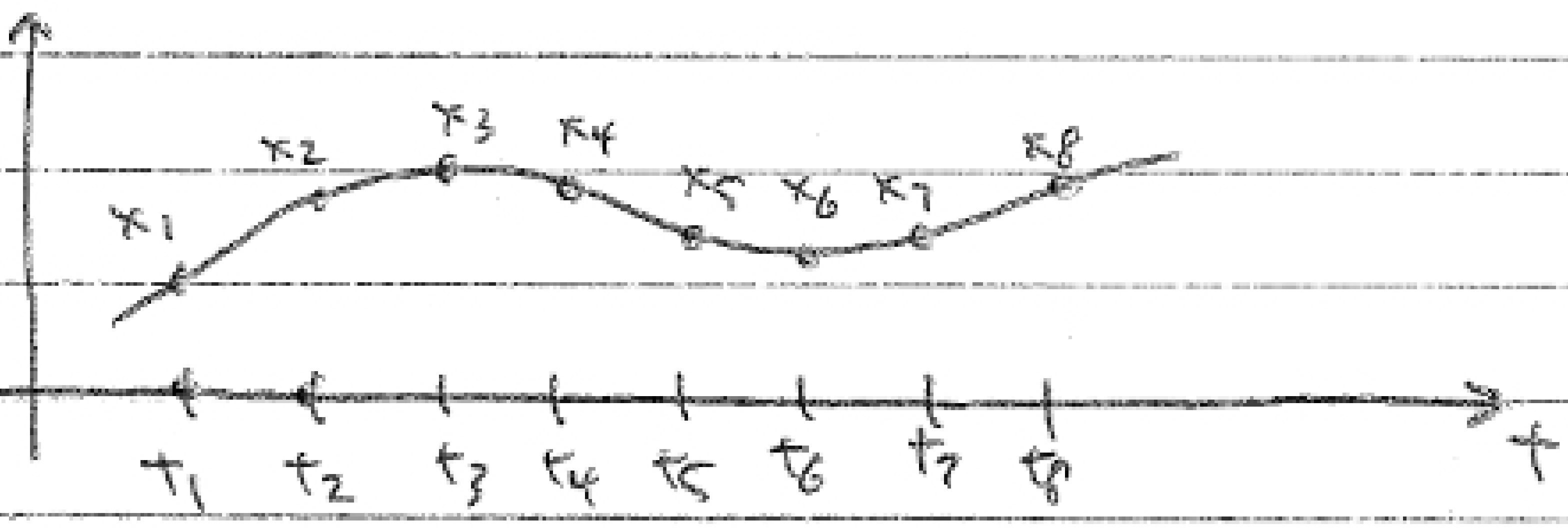
d. Plasma Simulation usually requires $y_i(t=0)$ as <u>initial conditions</u>.

I. (Continued)

B. Euler's Method:

1. To solve an equation numerically, we must represent it discretely. Take $\frac{dx}{dt} = v$

a.



b. Take $t_j \equiv j \Delta t$ and $x(t_j) \equiv x_j$ [and $v_j \equiv v(t_j)$]

2. Discretize $\boxed{\frac{dx}{dt} = \frac{x_{j+1} - x_j}{\Delta t} = v_j}$

a. Thus $\boxed{x_{j+1} = x_j + \Delta t\, v_j}$

3. Order of Convergence

a. Take $\frac{dy}{dt}\Big|_{t_j} = \frac{y_{j+1} - y_j}{\Delta t} \Rightarrow y_{j+1} = y_j + \Delta t \frac{dy}{dt}\Big|_{t_j}$

b. The true solution $y(t_{j+1}) = y(t_j) + \Delta t \frac{dy}{dt}\Big|_{t_j} + \frac{\Delta t^2}{2!}\frac{d^2y}{dt^2}\Big|_{t_j} + \frac{\Delta t^3}{3!}\frac{d^3y}{dt^3}\Big|_{t_j} + \dots$

c. Error $= y(t_{j+1}) - y_{j+1} = + \frac{\Delta t^2}{2!}\frac{d^2y}{dt^2}\Big|_{t_j} + \dots = \mathcal{O}(\Delta t^2)$

d. DEFINE: A method is order n if error scales as $\mathcal{O}((\Delta t)^{n+1})$.

$\boxed{\text{Thus, Euler's Method is first order.}}$

C. Higher Order Methods

1. Can we construct a higher order method for $\frac{dy}{dt}$?

Lecture #8 (Continued)

I.C. (Continued)

2. Take $\frac{dy}{dt}\Big|_{t_j} = \frac{y_{j+1} - y_{j-1}}{2\Delta t}$

3. In this case, $y_{j+1} = y_{j-1} + 2\Delta t \frac{dy}{dt}\Big|_{t_j}$

a. ~~Smthe~~ Take $y_{j-1} = y(t_j) - \Delta t \frac{dy}{dt}\Big|_{t_j} + \frac{\Delta t^2}{2!}\frac{d^2y}{dt^2}\Big|_{t_j} - \frac{\Delta t^3}{3!}\frac{d^3y}{dt^3}\Big|_{t_j} + \dots$

b. Thus $y(t_{j+1}) - y_{j+1} =$

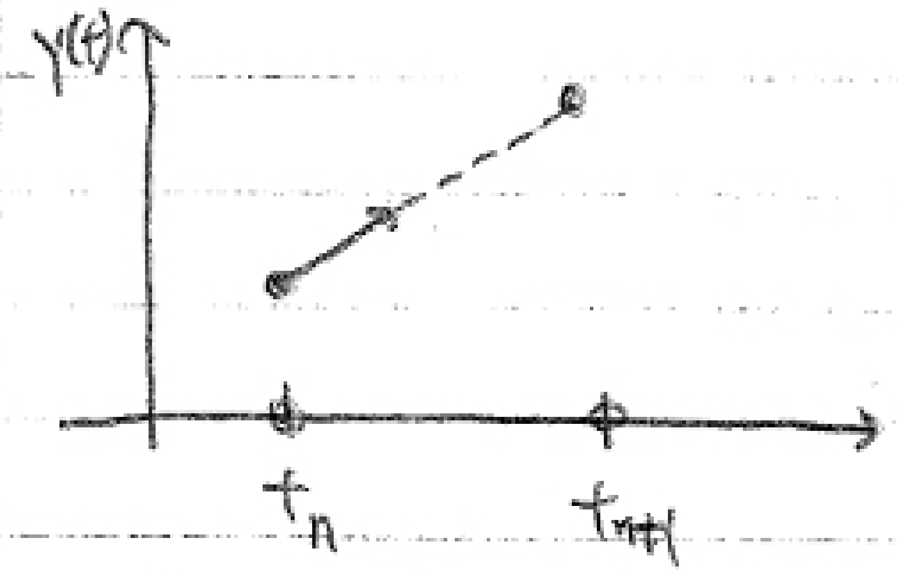
$$\cancel{y_j} + \cancel{\Delta t \frac{dy}{dt}\Big|_j} + \cancel{\frac{\Delta t^2}{2}\frac{d^2y}{dt^2}\Big|_j} + \frac{\Delta t^3}{3!}\frac{d^3y}{dt^3}\Big|_j - \left[\cancel{y_j} - \cancel{\Delta t\frac{dy}{dt}\Big|_j} + \cancel{\frac{\Delta t^2}{2!}\frac{d^2y}{dt^2}\Big|_j} - \frac{\Delta t^3}{3!}\frac{d^3y}{dt^3}\Big|_j + \dots\right]$$

$$- \cancel{2\Delta t \frac{dy}{dt}\Big|_j} = \frac{2\Delta t^3}{3!}\frac{d^3y}{dt^3}\Big|_j + \dots = O(\Delta t^3)$$

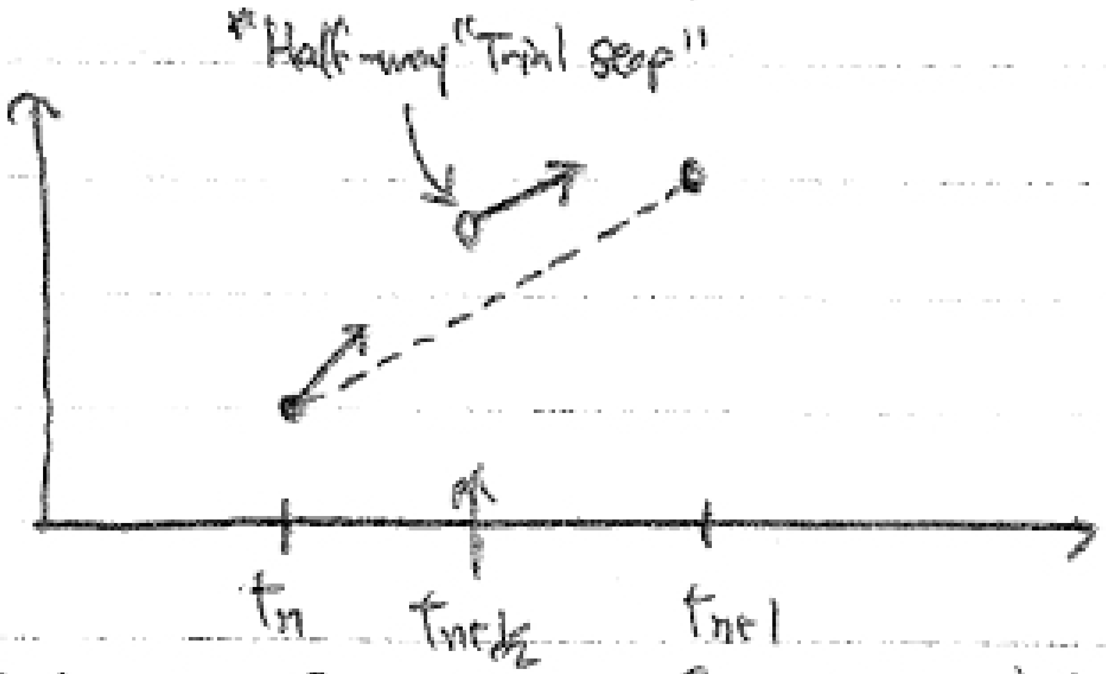
c. $\boxed{\text{Thus, this Centered Time difference is Second order}}$

D. Runge-Kutta Methods

1. Runge-Kutta Methods are a widely used workhorse of numerical methods.
2. In this multistep method, a "trial step" is taken to determine the derivative at the midpoint of the interval.



Euler's Method



2nd-order Runge-Kutta (or Midpoint) Method

3. The Method can be written:

$$\frac{dy}{dt} = f \Rightarrow$$

$$k_1 = \Delta t\, f(t_n, y_n)$$

$$k_2 = \Delta t\, f(t_{n+\frac{1}{2}}, y_n + \tfrac{1}{2}k_1)$$

$$y_{n+1} = y_n + k_2 + O(\Delta t^3)$$