Chapter 10

Linear Operators, Eigenvalues, and Green's Operator

We begin with a reminder of facts which should be known from previous courses.

10.1 Inner Product Space

A *vector space* V is a collection of objects $\{x\}$ for which addition is defined. That is, if $x, y \in V$, $x + y \in V$, which addition satisfies the usual commutative and associative properties of addition:

$$x + y = y + x, \quad x + (y + z) = (x + y) + z. \tag{10.1}$$

There is a zero vector 0, with the property

$$0 + x = x + 0 = x, \tag{10.2}$$

and the inverse of x, denoted $-x$, has the property

$$x - x \equiv x + (-x) = 0. \tag{10.3}$$

Vectors may be multiplied by complex numbers ("scalars") in the usual way. That is, if λ is a complex number, and $x \in V$, then $\lambda x \in V$. Multiplication by scalars is distributive over addition:

$$\lambda(x + y) = \lambda x + \lambda y. \tag{10.4}$$

Scalar multiplication is also associative: If λ and μ are two complex numbers,

$$\lambda(\mu x) = (\lambda\mu)x. \tag{10.5}$$

An *inner product space* is a vector space possessing an inner product. If x and y are two vectors, the inner product

$$\langle x, y\rangle \tag{10.6}$$

is a complex number. The inner product has the following properties:

$$\begin{aligned}
\langle x, y+\alpha z\rangle &= \langle x, y\rangle + \alpha\langle x, z\rangle, && (10.7a)\\
\langle x+\beta y, z\rangle &= \langle x, z\rangle + \beta^*\langle y, z\rangle, && (10.7b)\\
\langle x, y\rangle &= \langle y, x\rangle^*, && (10.7c)\\
\langle x, x\rangle &> 0 \quad \text{if} \quad x \neq 0, && (10.7d)
\end{aligned}$$

where α and β are scalars. Because of the properties (10.7a) and (10.7b), we say that the inner product is linear in the second factor and antilinear in the first. Because of the last property (10.7d), we define the *norm* of the vector by

$$\|x\| = \sqrt{\langle x, x\rangle}. \tag{10.8}$$

10.2 The Cauchy-Schwarz Inequality

An important result is the Cauchy-Schwarz inequality,[1] which has an obvious meaning for, say, three-dimensional vectors. It reads, for any two vectors x and y

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \tag{10.9}$$

where equality holds if and only if x and y are linearly dependent.

Proof: For arbitrary λ we have

$$0 \leq \langle x-\lambda y, x-\lambda y\rangle = \|x\|^2 - \lambda\langle x, y\rangle - \lambda^*\langle y, x\rangle + |\lambda|^2\|y\|^2. \tag{10.10}$$

Because the inequality is trivial if $y = 0$, we may assume $y \neq 0$, and so we may choose

$$\lambda = \frac{\langle y, x\rangle}{\|y\|^2}. \tag{10.11}$$

The the inequality (10.10) read

$$\begin{aligned}
0 \leq\; & \|x\|^2 - \frac{2}{\|y\|^2}|\langle x, y\rangle|^2 + \frac{|\langle y, x\rangle|^2}{\|y\|^2}\\
&= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}, && (10.12)
\end{aligned}$$

from which Eq. (10.9) follows. Evidently inequality holds in Eq. (10.10) unless

$$x = \lambda y. \tag{10.13}$$

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[1] The name Bunyakowskii should also be added.

From the Cauchy-Schwarz inequality, the triangle inequality follows:

$$\|x+y\| \leq \|x\| + \|y\|. \tag{10.14}$$

Proof:

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\,\langle x,y\rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2. \end{aligned} \tag{10.15}$$

QED

10.3 Hilbert Space

A *Hilbert space* $\mathcal{H}$ is an inner product space that is *complete*. Recall from Chapter 2 that a complete space is one in which any Cauchy sequence of vectors has a limit in the space. That is, if we have a Cauchy sequence of vectors, i.e., for any $\epsilon > 0$,

$$\{x_n\}_{n=1}^{\infty}: \quad \|x_n - x_m\| < \epsilon \quad \forall\, n,m > N(\epsilon), \tag{10.16}$$

then the sequence has a limit in $\mathcal{H}$, that is, there is an $x \in \mathcal{H}$ for which for any $\epsilon > 0$ there is an $N(\epsilon)$ so large that

$$\|x - x_n\| < \epsilon \quad \forall\, n > N(\epsilon). \tag{10.17}$$

We will mostly be talking about Hilbert spaces in the following.

Suppose we have a countable set of orthonormal vectors $\{e_i\}$, $i = 1, 2, \ldots$, in $\mathcal{H}$. Orthonormality means

$$\langle e_i, e_j\rangle = \delta_{ij}. \tag{10.18}$$

The set is said to be *complete* if any vector x in $\mathcal{H}$ can be expanded in terms of the e_is:[2]

$$x = \sum_{i=1}^{\infty} \langle e_i, x\rangle e_i. \tag{10.19}$$

Here convergence is defined in the sense of the norm as described above. Geometrically, the inner product $\langle e_i, x\rangle$ is a kind of direction cosine of the vector x, or a projection of the vector x on the basis vector e_i.

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[2] If the space is finite dimensional, then the sum runs up to the dimensionality of the space.