CS216: Program and Data Representation
University of Virginia Computer Science
Spring 2006 David Evans

Lecture 2: Orders of Growth

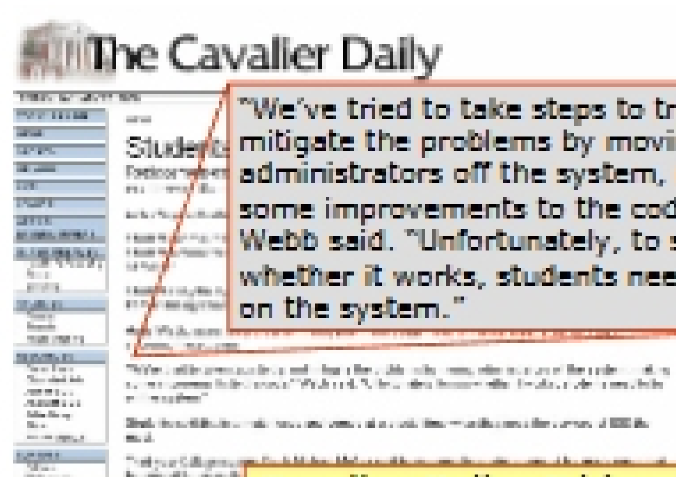
http://www.cs.virginia.edu/cs216

Menu

- Predicting program properties
- Orders of Growth: O, Ω
- Course Survey Results

Everyone should have received an email:
1. Informing you of your PS1 partner
2. Giving the section room locations
3. Explaining that PS1 is now due Monday, Jan 30

Predicting Program Properties



"We've tried to take steps to try and mitigate the problems by moving administrators off the system, making some improvements to the code," Webb said. "Unfortunately, to see whether it works, students need to be on the system."

Cavalier Daily, Friday Jan 20
(http://www.cavalierdaily.com/CVArticle.asp?ID=25461)

Predicting Running Time

- Experimentally: measure how long the program takes on particular inputs
  - Generalize, extrapolate to other input sizes
- Analytically: figure out how the amount of work scales with the input size
  - Understand what the program does
- Need to combine with experimental results on some inputs and analytical understanding to make good predictions

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Order Notation

- Four notations: $O(f)$, $\Omega(f)$, $o(f)$, $\Theta(f)$
- These notations define **sets of functions**
  - Functions from positive integer to real
- When we say, "Algorithm A is $O(n)$" we mean,

  running time of A $\in O(n)$

  where n measures the input size to A.

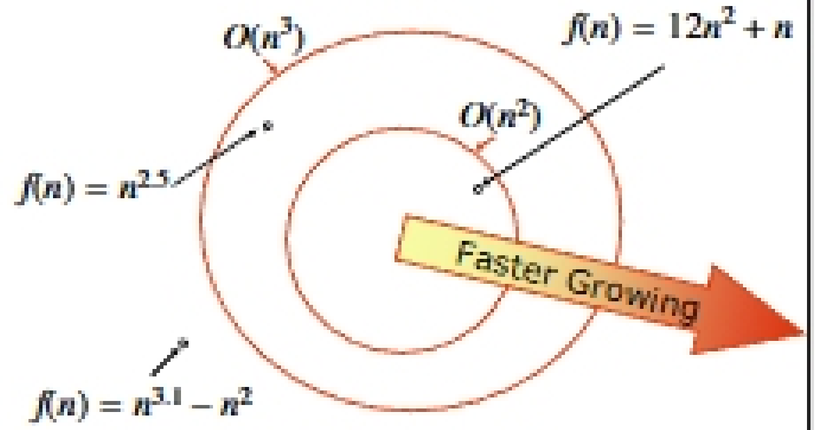
Big O

- Intuition: the set $O(f)$ is the set of functions that *grow* **no faster than** f
  - More formal definition coming soon
- Asymptotic growth rate
  - As input to f approaches infinity, how fast does value of f increase
  - Hence, only the fastest-growing term in f matters:

$O(n^2) \subset O(12n^2 + n)$

$O(n) = O(63n + \log n - 423)$

Examples



Formal Definition

$f \in O(g)$ means:

There are *positive* constants c and n_0 such that

$f(n) \leq cg(n)$

for all values $n \geq n_0$.

O Examples

$f(n) \in O(g(n))$ means: there are positive constants c and n_0 such that $f(n) \leq cg(n)$ for all values $n \geq n_0$.

$x \in O(x^2)$?	Yes, $c = 1$, $n_0 = 2$ works fine.
$10x \in O(x)$?	Yes, $c = 11$, $n_0 = 2$ works fine.
~~$x^2 \in O(x)$?~~	No, no matter what c and n_0 we pick, $cx^2 > x$ for big enough x

Question

Given $f \in O(h)$ and $g \notin O(h)$ which of these are true:

a. For **all** positive integers m, $f(m) < g(m)$.

b. For **some** positive integer m, $f(m) < g(m)$.

c. For some positive integer m_0, and all positive integers $m > m_0$, $f(m) < g(m)$. (left as problem for Exam 1)

a is false: Prove by Counter-Example

$f(n) \in O(h(n))$ and $g(n) \notin O(h(n))$

a. For all positive integers m, $f(m) < g(m)$.

Pick $h(n) = n^2$, $f(n) = 5n^2$, $g(n) = n^3$.

For $m = 2$, $f(m) = 20 > 8 = g(m)$.

Therefore, a is false.

> $f(n) \in O(g(n))$ means there are *positive* constants c and n_0 such that $f(n) \leq cg(n)$ for all values $n \geq n_0$.

b is true: Intuition

If $f \in O(h)$ and $g \notin O(h)$ then, for **some** positive integer m, $f(m) < g(m)$.

g must grow faster than h, otherwise g would be in $O(h)$.

f must grow no faster than h, since $f \in O(h)$

So, if g grows faster than h, but f grows as slow or slower than h, eventually, $g(n) > f(n)$ so for some m, $f(m) < g(m)$.

b: Proof by Contradiction

If $f \in O(h)$ and $g \notin O(h)$ then, for **some** positive integer m, $f(m) < g(m)$.

1. $f \in O(h) \Rightarrow$ there are *positive* constants c and n_0 such that $f(n) \leq ch(n)$ for all values $n \geq n_0$
2. $g \notin O(h) \Rightarrow$ there are **no** *positive* constants c_1 and n_1 such that $g(n) \leq c_1h(n)$ for all values $n \geq n_1$. So, for **all** *positive* constants c_2, $g(q) \leq c_2h(q)$ for some value q.

b: Proof by Contradiction

If $f \in O(h)$ and $g \notin O(h)$ then, for **some** positive integer m, $f(m) < g(m)$.

Suppose statement is false.

Then, for all positive k, $f(k) \geq g(k)$

From (1), $\exists c\ \exists n_0$ such that $\forall\ n > n_0, f(n) \leq ch(n)$

From (2), $\forall c_1\ \exists n_1$ such that $\forall\ n > n_1, g(n) > c_1h(n)$

Combining, $\exists c\ \forall c_1\ \exists n_2$ such that $\forall\ n > n_2$

$ch(n) > c_1h(n)$ Note: $n_2 \leq \max(n_0, n_1)$

This is a contradiction! Only works if $c = \text{infinity}$, but c must be a positive integer

Lower Bound: Ω (Omega)

$f(n)$ is $\Omega(g(n))$ means:

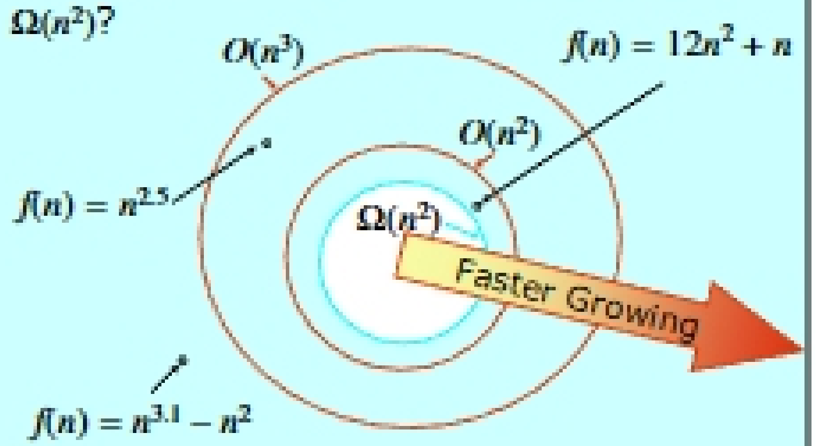
There are positive constants c and such that

$f(n) \geq cg(n)$

for all $n \geq n_0$.

Difference from O – this was $\leq$

Where is $\Omega(n^2)$?



Inside-Out