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MATH 2339. HW12: Solution

§10.8

28/ (3)

$\vec{r}(t) = \langle t^2, \ln t, t\ln t \rangle$, $(1,0,0)$ corresponds to $t=1$

$\boxed{\vec{r}'(t)} = \langle 2t, \frac{1}{t}, t\frac{1}{t} + \ln t \rangle \Rightarrow \vec{r}'(1) = \langle 2, 1, 1 \rangle$

$\boxed{\vec{r}''(t)} = \langle 2, -t^{-2}, \frac{1}{t} \rangle \Rightarrow \vec{r}''(1) = \langle 2, -1, 1 \rangle$

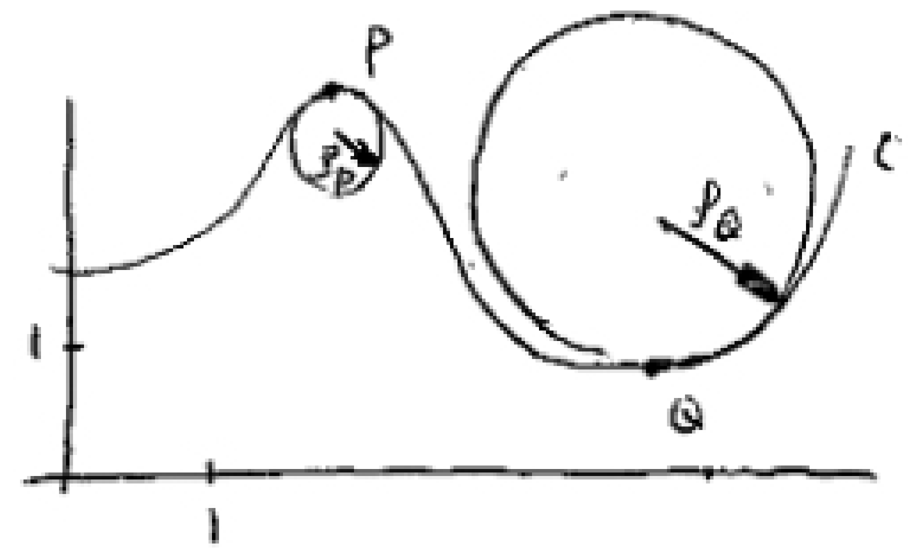
So $\boxed{\vec{r}'(1) \times \vec{r}''(1)} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 2\vec{i} - 4\vec{k}$

$$\kappa(1) = \frac{|\vec{r}'(1) \times \vec{r}''(1)|}{|\vec{r}'(1)|^3} = \frac{\sqrt{2^2 + (-4)^2}}{\left(\sqrt{2^2+1^2+1^2}\right)^3} = \frac{\sqrt{20}}{6\sqrt{6}} = \boxed{\frac{\sqrt{30}}{18}}$$

29/ (2)

(a) $\boxed{\kappa(P) > \kappa(Q)}$ because the curve turns quicker at P than at Q.

(b)



Based on the left figure:

$\kappa(P) = 1/\rho_P \approx \frac{1}{0.5} = 2$

$\kappa(Q) = 1/\rho_Q \approx \frac{1}{1} = 1$

(3) 40/

$\vec{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$ @ $(1,0,0)$ (@ $t = 2n\pi$)

$\boxed{\vec{r}'(t)} = \langle -\sin t, \cos t, \frac{1}{\cos t}(-\sin t) \rangle = \langle -\sin t, \cos t, -\tan t \rangle$

$$\boxed{\vec{T}(t)} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -\sin t, \cos t, -\tan t \rangle}{\sqrt{\underbrace{\sin^2 t + \cos^2 t}_{=1} + \tan^2 t}} = |\cos t| \langle -\sin t, \cos t, -\tan t \rangle$$

$(1 + \tan^2 t = \sec^2 t)$

for t near $2n\pi$, $|\cos t| = \cos t$

So: $\vec{T}(t) = \cos t \langle -\sin t, \cos t, -\tan t \rangle = \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle$

$\boxed{\vec{T}'(t)} = \langle \sin t \sin t - \cos t \cos t, \; 2\cos t(-\sin t), \; -\cos t \rangle$

Finally: $\vec{r}'(2n\pi) = \langle 0, 1, 0 \rangle$

$\vec{T}'(2n\pi) = \langle -1, 0, -1 \rangle$

We have $\vec{T} = \dfrac{\vec{r}'}{|\vec{r}'|} = \dfrac{\langle 0,1,0\rangle}{1} = \boxed{\langle 0, 1, 0 \rangle}$ ①

$\vec{N} = \dfrac{\vec{T}'}{|\vec{T}'|} = \dfrac{\langle -1, 0, -1\rangle}{\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}\langle -1, 0, -1 \rangle}$ ①

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{k} = \boxed{\left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle}$$ ①

⑤ 41/ $x = 2\sin 3t$, $y = t$, $z = 2\cos 3t$ @ $(0, \pi, -2)$ $(t = \pi)$

$\vec{r}(t) = \langle x, y, z \rangle = \langle 2\sin 3t, t, 2\cos 3t \rangle$

$\vec{r}'(t) = \langle 6\cos 3t, 1, -6\sin 3t \rangle$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle 6\cos 3t, 1, -6\sin 3t\rangle}{\sqrt{36\cos^2 3t + 1 + 36\sin^2 3t}} = \frac{1}{\sqrt{37}}\langle 6\cos 3t, 1, -6\sin 3t \rangle$$

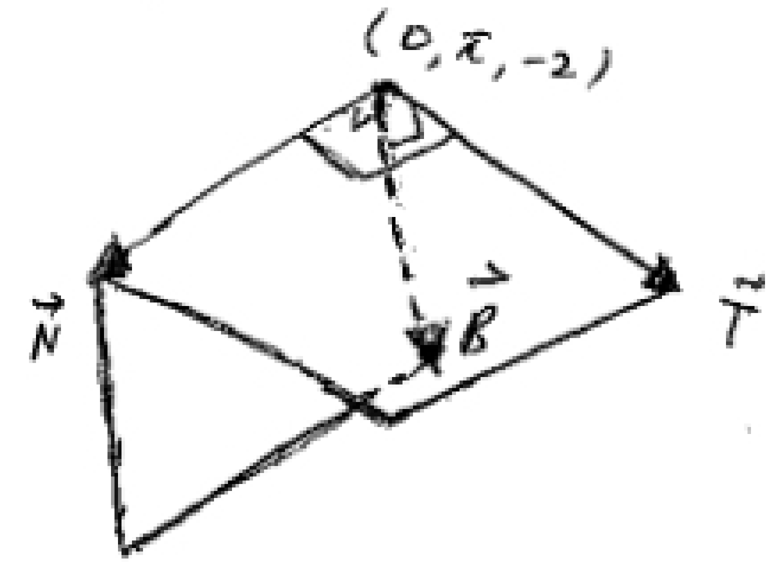
$\vec{T}'(t) = \frac{1}{\sqrt{37}} \langle -18\sin 3t, 0, -18\cos 3t \rangle$

So: ① $\boxed{\vec{T}(\pi)} = \frac{1}{\sqrt{37}}\langle 6\cos 3\pi, 1, -6\sin 3\pi \rangle = \frac{1}{\sqrt{37}}\langle -6, 1, 0 \rangle$

① $\boxed{\vec{N}(\pi)} = \dfrac{\vec{T}'(\pi)}{|\vec{T}'(\pi)|} = \dfrac{\frac{1}{\sqrt{37}}\langle 0, 0, 18 \rangle}{(18/\sqrt{37})} = \langle 0, 0, 1 \rangle$

① $\boxed{\vec{B}(\pi)} = \vec{T}(\pi) \times \vec{N}(\pi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{6}{\sqrt{37}} & \frac{1}{\sqrt{37}} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{37}}\vec{i} + \frac{6}{\sqrt{37}}\vec{j} = \langle \frac{1}{\sqrt{37}}, \frac{+6}{\sqrt{37}}, 0 \rangle$

The normal plane is formed by $\vec{B}$ and $\vec{N}$,
its normal is $\vec{T}$, so its eqn is:



$$\boxed{\begin{array}{l} -\frac{6}{\sqrt{37}}(x-0) + \frac{1}{\sqrt{37}}(y-\pi) = 0 \\ (\text{or } -6x + y - \pi = 0\ ) \end{array}}$$ ①

The oscalating plane is formed by $\vec{T}$
and $\vec{N}$, its normal is $\vec{B}$, so its eqn is:

$$\boxed{\begin{array}{l} \frac{1}{\sqrt{37}}(x-0) + \frac{6}{\sqrt{37}}(y-\pi) = 0 \\ (\text{or } \quad x + 6(y-\pi) = 0\ ) \end{array}}$$ ①

47/. ①

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$\left.\begin{array}{l} \boxed{\frac{d\vec{T}}{ds} = \frac{\vec{T}'(t)}{|\vec{r}'(t)|}} \\ \boxed{\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}} \end{array}\right\} \Rightarrow \boxed{\frac{d\vec{T}}{ds} = \frac{|\vec{T}'(t)|\,\vec{N}}{|\vec{r}'(t)|} = \kappa \vec{N}}$$