Homework #12 MATH 309 Solutions

Problem 1 Solve the following one-dimensional wave equation

$$u_{tt} = 4u_{xx}, \quad 0<x<3, \quad t>0$$

with the initial conditions

$$u(x,0) = x^2, \quad u_t(x,0) = 0$$

and the boundary conditions

$$u(0,t) = u(3,t) = 0, \quad t>0$$

Solution First find solutions in the form

$$u(x,t) = X(x)T(t)$$

Substituting into equation:

$$XT'' = 4X''T \Rightarrow \frac{X''}{X} = \frac{T''}{4T} = c \Rightarrow$$

$$X'' - cX = 0$$
$$T'' - 4cT = 0$$

$u(0,t) = u(3,T) = 0 \Rightarrow X(0) = X(3) = 0 \Rightarrow$ by analogy with what we did previously for the heat equation and the Laplace equation $c = -\frac{\pi n^2}{L^2} = -\frac{\pi^2 n^2}{9}, n \in N$ (here $L=3$) and

$$X(x) = B \sin \frac{\pi n}{3} x$$

Analyze now the equation for T.

$$T'' + \frac{4\pi^2 n^2}{9} T = 0 \Rightarrow T(t) = C\cos\frac{2\pi n}{3}t + D\sin\frac{2\pi n}{3}t$$

The condition $u_t(x,0) = 0$ implies that $T'(0) = 0 \Rightarrow$

$$-C\frac{2\pi n}{3}\sin\frac{2\pi n}{3}t + \frac{2\pi n}{3}D\cos\frac{2\pi n}{3}t\Big|_{t=0} = \frac{2\pi n}{3}D = 0 \Rightarrow$$

$p=0 \Rightarrow T(t)=C\cos\frac{2\pi n}{3}t \Rightarrow$

$$u(x,t)=b_n\cos\frac{2\pi n}{3}t\,\sin\frac{\pi n}{3}x$$

We seek for the solution satisfying $u(x,0)=x^2$ in the form

$$u(x,t)=\sum_{n=1}^{\infty} b_n\cos\frac{2\pi n}{3}t\,\sin\frac{\pi n}{3}x \qquad (*)$$

So $u(x,0)=\sum_{n=1}^{\infty} b_n\sin\frac{\pi n}{3}x = x^2 \qquad 0<x<3$

Therefore to find coefficients b_n we have to expand x^2 in the <u>Fourier sine</u> series on the interval $(0,3)$

$$b_n=\frac{2}{3}\int_0^3 x^2\sin\frac{\pi n}{3}x\,dx=\frac{2}{3}\cdot-\frac{x^2\cos\frac{\pi n}{3}x}{\frac{\pi n}{3}}\Big|_0^3+\frac{2}{3}\int_0^3\frac{2x\cos\frac{\pi n}{3}x}{\frac{\pi n}{3}}dx=$$

$$-\frac{2}{\pi n}x^2\cos\frac{\pi n}{3}x\Big|_0^3+\frac{4}{\pi n}\int_0^3 x\cos\left(\frac{\pi n}{3}x\right)dx=$$

$$=-\frac{2}{\pi n}9\cos\pi n+\frac{4}{\pi n}\frac{x\sin\frac{\pi n}{3}x}{\frac{\pi n}{3}}\Big|_0^3-\frac{4}{\pi n}\int_0^3\frac{\sin\frac{\pi n}{3}x}{\frac{\pi n}{3}}dx=$$

$$=-\frac{18}{\pi n}(-1)^n+\frac{12}{\pi^2n^2}\frac{\cos\frac{\pi n}{3}x}{\frac{\pi n}{3}}\Big|_0^3=\frac{18}{\pi n}(-1)^{n+1}+\frac{36}{\pi^3n^3}(\cos\pi n-1)=$$

$$=\underbrace{\frac{18}{\pi n}(-1)^{n+1}+\frac{36}{\pi^3n^3}((-1)^n-1)}_{=\frac{18}{\pi n}\left((-1)^{n+1}+\frac{2}{\pi^2n^2}((-1)^n-1)\right)}=\begin{cases}\frac{18}{\pi n}-\frac{72}{\pi^3n^3} & \text{if } n \text{ is odd}\\ -\frac{18}{\pi n} & \text{if } n \text{ is even}\end{cases}$$

substituting to $(*)$

$\Rightarrow$

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}\frac{18}{\pi n}\left((-1)^{n+1}+\frac{2}{\pi^2n^2}((-1)^n-1)\right)\cos\frac{2\pi n}{3}t\,\sin\frac{\pi n}{3}x}$$

Problem2 Solve the following boundary value

$$u_{xx}+u_{yy}=0 \text{ for } x^2+y^2<1$$

$$u(x,y)=y^3 \quad x^2+y^2=1$$

Solution Passing to polar coordinates $x=r\cos\theta$, $y=r\sin\theta$

we get

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

$$u(1,\theta)=\sin^3\theta \quad , u(r,\theta) \text{ is periodic w.r.t. } \theta \text{ with period } 2\pi$$

$u(0,\theta)$ is finite.

Seperate the variables $u(r,\theta)=R(r)\Theta(\theta)$.

Substiduting into equation

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0 \Rightarrow \text{ dividing by } R\Theta$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Theta''}{\Theta}=0 \Rightarrow$$

$$r^2\left(\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}\right)=-\frac{\Theta''}{\Theta}=C$$

$$\begin{cases} \dfrac{r^2R''+rR'}{R}=C & (1) \\ \Theta''+C\Theta=0 & (2) \end{cases}$$

$\Theta''+C\Theta=0$ with $\Theta(\theta)$ being periodic with period 2π

$\Rightarrow C=n^2$, $n\geq 0$, n is integer (see lecture #7, class of April 26 for more details on this) $\Rightarrow \Theta''+n^2\Theta=0 \Rightarrow$

$$\Theta(\theta)=a_n\cos n\theta+b_n\sin n\theta \text{ if } n>0 \text{ or } \Theta(\theta)\equiv \text{const} \text{ in } n=0$$

Substiduting $C=n^2$ into (1) we get the Euler equation