MATH 2339, HW13: Solution (14)

§10.9

7/. (3)

$\vec{r}(t) = \langle t^2+t,\ t^2-t,\ t^3 \rangle$

$\boxed{\vec{v}(t)} = \vec{r}\,'(t) \boxed{= \langle 2t+1,\ 2t-1,\ 3t^2 \rangle}$ (1)

$\boxed{\vec{a}(t)} = \vec{v}\,'(t) = \boxed{\langle 2,\ 2,\ 6t \rangle}$ (1)

$\boxed{v(t)} = |\vec{v}(t)| = \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2}$

$= \sqrt{(4t^2+4t+1) + (4t^2-4t+1) + 9t^4}$

$\boxed{= \sqrt{8t^2+2+9t^4}}$ (1)

8/. (2)

$\vec{r}(t) = \langle 2\cos t,\ 3t,\ 2\sin t \rangle$

$\vec{v}(t) = \vec{r}\,'(t) = \langle -2\sin t,\ 3,\ 2\cos t \rangle$ (1)

$\vec{a}(t) = \vec{v}\,'(t) = \langle -2\cos t,\ 0,\ -2\sin t \rangle$

$\boxed{v(t)} = |\vec{v}(t)| = \sqrt{(-2\sin t)^2 + 3^2 + (2\cos t)^2}$

$= \sqrt{4(\sin^2 t + \cos^2 t) + 9} \boxed{= \sqrt{13}}$ (1)

12/. (3)

$\vec{a}(t) = 2\vec{i} + 6t\vec{j} + 12t^2\vec{k}$, $\vec{v}(0) = \vec{i}$, $\vec{r}(0) = \vec{j} - \vec{k}$

(1) $\vec{v}(t) = \int \vec{a}(t)\,dt = (2t + c_1)\vec{i} + (3t^2 + c_2)\vec{j} + (4t^3 + c_3)\vec{k}$

$\vec{v}(0) = \vec{i} \Rightarrow c_1\vec{i} + c_2\vec{j} + c_3\vec{k} \equiv \vec{c} = \vec{i} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$ (1)

So $\boxed{\vec{v}(t) = (2t+1)\vec{i} + (3t^2)\vec{j} + (4t^3)\vec{k}}$ (1)

(2) $\vec{r}(t) = \int \vec{v}(t)\,dt = (t^2+t+d_1)\vec{i} + (t^3+d_2)\vec{j} + (t^4+d_3)\vec{k}$

$\vec{r}(0) = \vec{j} - \vec{k} \Rightarrow d_1\vec{i} + d_2\vec{j} + d_3\vec{k} = \vec{d} = \vec{j} - \vec{k} \Rightarrow \begin{cases} d_1 = 0 \\ d_2 = 1 \\ d_3 = -1 \end{cases}$ (1)

So $\boxed{\vec{r}(t) = (t^2+t)\vec{i} + (t^3+1)\vec{j} + (t^4-1)\vec{k}}$ (1)

31/. ④

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$$

$$\vec{a}(t) = a_T\vec{T} + a_N\vec{N}$$

where $a_T = v' = \dfrac{\vec{v}\cdot\vec{a}}{v} = \dfrac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|}$

$$a_N = \kappa v^2 = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|}$$

STEPS ②

$$\vec{r}'(t) = \langle -\sin t,\ \cos t,\ 1\rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\vec{r}''(t) = \langle -\cos t,\ -\sin t,\ 0\rangle$$

$$\vec{r}'\cdot\vec{r}'' = \sin t\cos t + (-\cos t\sin t) + 0 = 0$$

$$\vec{r}'\times\vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \sin t\,\vec{i} - (\cos t)\,\vec{j} + (\sin^2 t + \cos^2 t)\,\vec{k}$$

$$= \langle \sin t,\ -\cos t,\ 1\rangle$$

$$|\vec{r}'\times\vec{r}''| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

So:

$$\boxed{a_T = \frac{\vec{r}'\cdot\vec{r}''}{|\vec{r}'|} = 0} \quad ①$$

$$\boxed{a_N = \frac{\sqrt{2}}{\sqrt{2}} = 1} \quad ①$$

34/. ②

$$\vec{L}(t) = m\vec{r}(t)\times\vec{v}(t) \Rightarrow \vec{L}'(t) = m\left[\vec{r}(t)\times\vec{v}'(t) + \vec{r}'(t)\times\vec{v}(t)\right]$$

$$= m\left[\vec{r}(t)\times\vec{a}(t) + \vec{v}(t)\times\vec{v}(t)\right]$$

$$= m\,\vec{r}(t)\times\vec{a}(t) = \vec{\tau}(t)$$

If $\vec{\tau}(t) = 0$, then $\vec{L}'(t) = 0 \Rightarrow \vec{L}(t) = \vec{c}$ (a constant vector)