
MATH 2300 - Calculus III

Fall 2008

Homework 2 - Solutions

For questions 1–7, assume that θ is a fixed number in the interval $[0, 2\pi)$ and the position $\mathbf{R}$ of a particle moving in space is given by

$$x = \cos t, \quad y = \cos\theta \sin t, \quad z = \sin\theta \sin t$$

for $-\infty < t < \infty$.

1. Show that the particle lies on the unit sphere (that is, the set $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$) for all values of t.

   SOLUTION: The position vector R has length one:

$$\begin{aligned} R \cdot R &= \cos^2 t + \cos^2\theta \sin^2 t + \sin^2\theta \sin^2 t \\ &= \cos^2 t + \sin^2 t(\sin^2\theta + \cos^2\theta) \\ &= 1. \end{aligned}$$

2. Determine the velocity, the speed and the acceleration of the particle.

   SOLUTION: The velocity vector at t is

$$\mathbf{R}'(t) = -\sin t\mathbf{i} + \cos\theta\cos t\mathbf{j} + \sin\theta\cos t\mathbf{k}.$$

   The speed is the length of the velocity vector. It's best to compute the square of the speed first:

$$\begin{aligned} \mathbf{R}'(t) \cdot \mathbf{R}'(t) &= \cos^2 t + \cos^2\theta \sin^2 t + \sin^2\theta \sin^2 t \\ &= \cos^2 t + \sin^2 t(\cos^2\theta + \sin^2\theta) \\ &= 1. \end{aligned}$$

   It is now clear that $|\mathbf{R}'(t)| = 1$. The acceleration vector at t is

$$\mathbf{R}''(t) = -\cos t\mathbf{i} - \cos\theta\sin t\mathbf{j} - \sin\theta\sin t\mathbf{k}.$$

3. Show that the velocity is tangent to the unit sphere at $\mathbf{R}$ for every value of t.

   SOLUTION: The curve $\mathbf{R}$ lies on the unit sphere, so the velocity vectors of this curve are tangent to the sphere by definition. Or, if you like, you can show that the dot product of the position vector and the velocity vector are orthogonal. In fact,

$$\begin{aligned} \mathbf{R}(t) \cdot \mathbf{R}'(t) &= -\cos t\sin t + \cos^2\theta\cos t\sin t + \sin^2\theta\cos t\sin t \\ &= -\cos t\sin t + \cos t\sin t(\cos^2\theta + \sin^2\theta) \\ &= 0. \end{aligned}$$

4. Find the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$ and the binormal $\mathbf{B}$ for every value of t.

   SOLUTION: The unit tangent vector is exactly the velocity vector because the speed is unity. So,

$$\mathbf{T}(t) = -\sin t\mathbf{i} + \cos\theta\cos t\mathbf{j} + \sin\theta\cos t\mathbf{k}.$$

   The normal vector is obtained in two steps. First calculate

$$\mathbf{T}'(t) = -\cos t\mathbf{i} - \cos\theta\sin t\mathbf{j} - \sin\theta\sin t\mathbf{k}.$$

Next it is easy show just like in the case of the velocity vector that this vector has length one. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\mathbf{i} - \cos\theta\sin t\mathbf{j} - \sin\theta\sin t\mathbf{k}.$$

The binormal is given by

$$\begin{aligned} \mathbf{B}(t) &:= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= -\sin\theta\mathbf{j} + \cos\theta\mathbf{k}. \end{aligned}$$

5. Is the binormal vector tangent to the sphere at $\mathbf{R}(t)$ for every value of t? Explain.

SOLUTION: The binormal is tangent to the sphere at $\mathbf{R}(t)$. The reason is simple: $\mathbf{B}(t)$ is orthogonal to the point $\mathbf{R}(t)$ (which lies on the sphere) for every value of t. Indeed,

$$\mathbf{R}(t) \cdot B(t) = 0.$$

6. Does the unit normal vector at $\mathbf{R}(t)$ point into the unbounded region of space bounded by the sphere or the bounded region bounded by the sphere? (In the first case we say $\mathbf{N}$ is the outer normal; in the second case it is called the inner normal.) Explain.

SOLUTION: It points into the bounded region; it is the inner normal. To see this, consider the position vector $\mathbf{R}(t)$. It points from the origin to the point which we also call $\mathbf{R}(t)$ on the sphere. Thus we can think of this vector as pointing toward the unbounded region. It is in the direction of the outer normal. The unit normal $\mathbf{N}(t)$ is parallel to the position vector but it points in the opposite direction. For example, we have that

$$\begin{aligned} \mathbf{R}(t) \cdot \mathbf{N}(t) &= -\cos^2 t - \cos^2\theta\sin^2 t - \sin^2\theta\sin^t \\ &= -(\cos^2 t + \sin^t(\sin^2\theta + \cos^2\theta)) \\ &= -1. \end{aligned}$$

The dot product is defined to be the length of $\mathbf{R}(t)$ times the length of $\mathbf{N}(t)$ times the cosine of the included angle. Both lengths are unity. So, the cosine of the angle is -1. This means the angle is π (or 180°).

7. Find the curvature of the parametric curve $t \mapsto R(t)$ at $R(t)$ for every value of t.

SOLUTION: The simplest way to compute the curvature is to use the formula

$$\kappa := \frac{|\mathbf{R}'(t) \times \mathbf{R}''(t)|}{(\mathbf{R}'(t) \cdot \mathbf{R}'(t))^{3/2}}.$$

In our case, since the length of the velocity vector is unity, we don't have to recompute the denominator. Let us simply compute

$$\begin{aligned} \mathbf{R}'(t) \times \mathbf{R}''(t) &= -\sin\theta(\cos^2 t + \sin^2 t)\mathbf{j} - \cos\theta(\cos^2 t + \sin^2 t)\mathbf{k} \\ &= -\sin\theta\mathbf{j} - \cos\theta\mathbf{k} \end{aligned}$$

and note that the length of this vector is unity. Thus, $\kappa = 1$.

8. Find the arc-length of the curve $t \mapsto R(t)$ on the interval $0 \le t \le 2\pi$.

SOLUTION: By now you should be getting the idea that the curve we are discussing is a circle! It lies in a plane (why) and its radius is one. Thus the arc-length once around the circle—because all functions defining it have period 2π—must be 2π. This is the desired arc-length. Using formulas, the arc-length is

$$\int_0^{2\pi} |R'(t)|\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

For questions 9–12, consider a bead with mass m sliding on a circular wire that we will idealize to be the circle in the xz-plane centered at the origin with radius $L > 0$. Ignore friction and assume that the only force acting on the bead is the force due to gravity, which is given by the vector $-gm\mathbf{k}$. We are imagining that the positive z-axis points up and the x-axis is parallel to the (flat) earth. Since the particle moves on this circle, its position can be given by parametric equations

$$x = L\sin(\theta(t)), \qquad z = -L\cos(\theta(t))$$

for some unknown function θ whose domain and range are the real numbers. For definiteness, assume that $\theta'(t) > 0$ for every t. Let $\mathbf{a}$ denote the acceleration of the particle.

9. Write the vector $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and unit normal corresponding to the motion.

SOLUTION: In this case, we have that

$$\begin{aligned}\mathbf{T}(t) &= \cos(\theta(t))\mathbf{j} + \sin(\theta(t))\mathbf{k}\\ \mathbf{N}(t) &= -\sin(\theta(t))\mathbf{j} + \cos(\theta(t))\mathbf{k}\\ \mathbf{a}(t) &= (L\theta''(t)\cos(\theta(t)) - L\theta'(t)^2\sin(\theta(t)))\mathbf{j} + (L\theta''(t)\sin(\theta(t)) + L\theta'(t)^2\sin(\theta(t)))\mathbf{k}.\end{aligned}$$

It should be clear that

$$\begin{aligned}a_T(t) &= \mathbf{a}(t).\mathbf{T}(t)\\ &= \cos(\theta(t))(L\theta''(t)\cos(\theta(t)) - L\theta'(t)^2\sin(\theta(t))) + \cos(\theta(t))(L\theta''(t)\sin(\theta(t)) + L\theta'(t)^2\sin(\theta(t)))\\ &= L\theta''(t),\\ a_N(t) &= L\theta'(t)^2;\end{aligned}$$

therefore,

$$\mathbf{a}(t) = L\theta''(t)\mathbf{T}(t) + L\theta'(t)^2\mathbf{N}(t).$$

10. Determine scalars α and β such that $-gm\mathbf{k} = \alpha\mathbf{T} + \beta\mathbf{N}$.

SOLUTION: Again, by taking the dot product of both sides of the equation first with respect to $\mathbf{T}$ *and then with* $\mathbf{N}$*, it follows that*

$$\begin{aligned}\alpha &= -gm\mathbf{k}\cdot\mathbf{T}(t) = -gm\sin(\theta(t)),\\ \beta &= -gm\mathbf{k}\cdot\mathbf{N}(t) = -gm\cos(\theta(t)).\end{aligned}$$

11. Newton's second law tells us that $m\mathbf{a} = F$, where F is the sum of all the forces acting on the bead. There are some very complicated (electrodynamic) forces that hold the bead on the wire. Let's assume that these forces act in the normal direction only. In this case, we need only consider the tangential component of the vector equation $m\mathbf{a} = F$ to determine the effect of gravity: $ma_T = \alpha$. Write this equation as an equation (which is called the equation of motion) for the unknown function θ.

The same equation is valid in case $\theta'(t) \le 0$ at some values of t; but, you are not asked to show this fact. It would be very nice to find all possible functions θ that solve the equation of motion; they could be used to predict the possible motions of the bead. But, to determine these functions, you will have to learn more very beautiful mathematics that is not part of this course.

SOLUTION: Since $ma_T = \alpha$ *and using our calculations, we have the equation*

$$mL\theta''(t) = -gm\sin(\theta(t)),$$

or, better yet, the equation

$$L\theta''(t) = -g\sin(\theta(t)),$$