
MATH 2300 - Calculus III

Fall 2008

Homework 3

DUE Wednesday, 8 October 2008

1. Find the equation for the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 1$ at the point $(1/\sqrt{3}, 1/\sqrt{6}, 1/3)$ in the form $Ax + By + z = D$.
2. Find the directional derivative of the function f given by $f(x, y, z) = ze^{xy}$ in the direction $(1/\sqrt{2}\mathbf{i} + 1/\sqrt{2}\mathbf{k})$ at the point $(1/2, \ln 2, 1/\ln 2)$.
3. Suppose that S is a surface in space that happens to be given as the graph of a continuously differentiable function f ; that is, $S = \{(x, y, f(x, y)) : x \text{ and } y \text{ are real numbers}\}$. While we don't know anything else about the function f , suppose that the curves given by the parametric formulas

$$\gamma(t) = (1 + t, 1 + t^2, 1 + t^3), \quad \sigma(t) = (e^t, e^{t/2}, e^{t/3})$$

both lie in S for all values of t in the interval $-1/100 < t < 1/50$. Find the equation of the tangent plane to S at the point $(1, 1, 1)$ in the form $Ax + By + Cz = 1$.

4. Find the minimum distance between the origin and the surface $z^2 = \frac{8 - y^2 - x^4y^2}{2}$.
5. Suppose that there is a twice continuously differentiable function $r = f(t, s)$ such that for every pair of real numbers (t, s) we have the identity

$$\frac{\partial f}{\partial t}(t, s) = \sin(f(t, s));$$

or, in other words,

$$\frac{\partial r}{\partial t} = \sin r.$$

Determine a similar identity that is satisfied by the function $q = \frac{\partial f}{\partial s}(t, s)$. Your answer might involve r . More specifically, one side of the requested identity should be $\frac{\partial q}{\partial t} =$. The other side should be a function of q and r and not contain the symbol ∂ .

6. (a) If $H(x, y, z) = xyz$ and $\boldsymbol{\eta} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show that the vector

$$\nabla H(x, y, z) - 3xyz\boldsymbol{\eta}$$

is tangent to the unit sphere at every point (x, y, z) that lies on the sphere.

(b) If $H(x, y, z)$ is a homogeneous polynomial function of degree three; that is, H is a polynomial in three variables and $H(\lambda x, \lambda y, \lambda z) = \lambda^3 H(x, y, z)$ for every number λ and $\boldsymbol{\eta} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show that the vector

$$\nabla H(x, y, z) - 3H(x, y, z)\boldsymbol{\eta}$$

is tangent to the unit sphere at every point (x, y, z) that lies on the sphere.

7. Suppose that Φ is a continuously differentiable function defined in all of space and \mathbf{r} defines a parametrized curve in space such that $\mathbf{r}'(t) = -\nabla\Phi(\mathbf{r}(t))$ for every value of the parameter t . Show that the composite function $t \mapsto \Phi(\mathbf{r}(t))$ is non increasing; that is, this function either decreases or stays constant as t increases.

8. Suppose that the total force acting on a particle with mass m moving in space with position vector \mathbf{r} is given by $-\nabla\Phi(\mathbf{r})$. According to Newton's second law of motion, the position must satisfy the equation $m\mathbf{r}'' = -\nabla\Phi(\mathbf{r})$. The kinetic energy of the particle is defined to be $\frac{1}{2}m\mathbf{r}' \cdot \mathbf{r}'$; and, in this case, the potential energy is defined to be Φ . The total energy is the sum of the potential and kinetic energies:

$$E := \frac{1}{2}m\mathbf{r}' \cdot \mathbf{r}' + \Phi(\mathbf{r}).$$

Show that the energy remains constant during the motion of the particle.

9. For each number E , define S_E to be the set

$$S_E := \{(x, y) : \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = E\}$$

Show that the vector $y\mathbf{i} + (x - x^3)\mathbf{j}$ is tangent to S_E at every point of S_E .

10. (a) Determine the critical points of the function F given by

$$F(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4.$$

and determine which are minima, maxima and saddles. (b) Does F have a global minimum? If so, specify this value and the points at which it is attained. Hint: One way to proceed is to change to polar coordinates so that F has the form $r^2(r^2 \cos^4 \theta - 4 \cos^2 \theta + 2)/4$. Note that if $r \geq 2$, then

$$r^2(r^2 \cos^4 \theta - 4 \cos^2 \theta + 2)/4 \geq 4 \cos^4 \theta - 4 \cos^2 \theta + 2.$$

One variable calculus (perhaps with the change of variables $z = \cos^2 \theta$) can be used to show $4 \cos^4 \theta - 4 \cos^2 \theta + 2 \geq 1$. Thus, on the circle of radius $r = 2$, F has minimum value 2 and for every point whose distance from the origin is greater than 2, F has minimum value larger than 1.