Homework assignment #3, Solutions, MATH309

Problem 1 (a)

$$\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix} \underset{\substack{\text{expansion}\\ \text{along the}\\ \text{first row}}}{=} -a \begin{vmatrix} b & d \\ 0 & 0 \end{vmatrix} = \boxed{0}$$

(b)

$$\begin{vmatrix} -10 & 4 & -1 & 6 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 7 & -3 & 2 & 8 \end{vmatrix} \underset{\substack{\text{expansion}\\ \text{along the}\\ \text{second row}}}{=} 3 \begin{vmatrix} -10 & -1 & 6 \\ 0 & 0 & -2 \\ 7 & 2 & 8 \end{vmatrix} \underset{\substack{\text{expansion}\\ \text{along the}\\ \text{second row}}}{=} 3\cdot(-(-2)) \begin{vmatrix} -10 & -1 \\ 7 & 2 \end{vmatrix} =$$

$$= 6\,(-20+7) = -6\cdot 13 = \boxed{-78}$$

(c)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} \underset{\substack{R_2\to R_2-R_1\\ R_3\to R_3-R_2}}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0$$ (because 2 rows coinside

Problem 2 (a)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -4 & 0 & 1 & 0 \\ 7 & 6 & -5 & 0 & 0 & 1 \end{array}\right) \underset{R_3\to R_3-7R_1}{\overset{R_2\to R_2-3R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & -8 & 2 & -7 & 0 & 1 \end{array}\right) \overset{R_3\to R_3+8R_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & -6 & -31 & 8 & 1 \end{array}\right) \underset{R_3\to -\frac{1}{6}R_3}{\overset{R_1\to R_1-2R_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 7 & -2 & 0 \\ 0 & 1 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{array}\right) \underset{R_2\to R_2+R_3}{\overset{R_1\to R_1-R_3}{\sim}}$$

The matrix A is invertible; Because the is no free variables

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7-\frac{31}{6} & -2+\frac{4}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -3+\frac{31}{6} & 1-\frac{4}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{6} & -\frac{2}{3} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{13}{6} & -\frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} \frac{11}{6} & -\frac{2}{3} & \frac{1}{6} \\ \frac{13}{6} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{31}{6} & -\frac{4}{3} & -\frac{1}{6} \end{pmatrix}$$

(b) $\det\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{pmatrix} \overset{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_2}}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow$

A is invertible

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 5 & 4 \\ 6 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 4 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 3 & 5 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 25-24 & -(5-6) & 4-5 \\ -(15-12) & 5-3 & -(4-3) \\ 18-15 & -(6-3) & 5-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix} \Rightarrow$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix}$$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{pmatrix}$

Using Jordan-Gauss reduction:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 7 & -10 & 0 & 1 & 0 \\ 7 & 16 & -21 & 0 & 0 & 1 \end{array}\right) \underset{R_3 \to R_3 - 7R_1}{\overset{R_2 \to R_2 - 3R_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 0 & 1 & 0 \\ 0 & 2 & -14 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\overset{R_3 \to R_3 - 2R_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 \end{array}\right) \longrightarrow$$

The matrix is not invertible

(for example because $Ax = 0$ has nontrivial solution)

Note that the same row operations show that $\det A = 0$

i.e. again A is not invertible

Problem 3

(a) For $a_{11} a_{22} a_{33} a_{44} a_{55} a_{66}$ the corresponding permutation is $(123456) \Rightarrow$ no inversion $\Rightarrow$ the sign is $\boxed{+}$

(b) The term $a_{12} a_{23} a_{35} a_{46} a_{52} a_{64}$ does not appear in the expansion of the determinant, because the entries a_{12} and a_{52} from the same column appear in this term,

because no entry from the first column appears in this term.

(c) For $a_{13} a_{24} a_{36} a_{42} a_{55} a_{61}$ the corresponding permutation is $(346251) \Rightarrow$ the inversions are

$(3,2), (3,1), (4,2), (4,1), (6,2), (6,5), (6,1), (2,1), (5,1) \Rightarrow$

we have 9 inversions $\Rightarrow$ it is an odd permutation $\Rightarrow$ the sign is $\boxed{\text{"}-\text{"}}$

Another way is to count the parity of the number of transpositions needed for transforming our permutation to the permutation (123456):