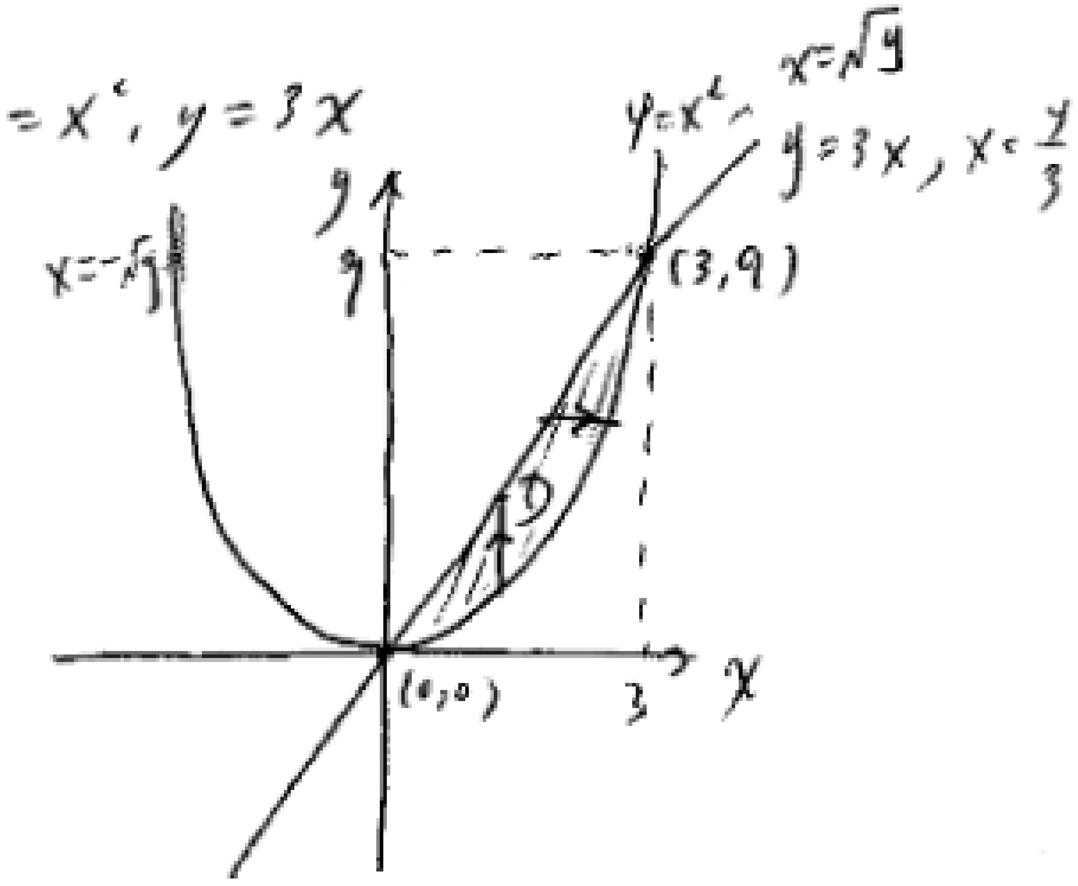
MATH 2339, HW30: Solution (22)

§12.2

(8) 12/. $\iint_D xy\, dA$, D: enclosed by $y=x^2, y=3x$



[Soln] $\begin{cases} y=x^2 \\ y=3x \end{cases} \Rightarrow x^2=3x \Rightarrow x(x-3)=0$

$\Rightarrow x=0, 3$

So $y=x^2$ and $y=3x$ intersect @ $(0,0)$ and $(3,9)$

• Type I: $D=\{(x,y) \mid 0\le x\le 3,\ x^2\le y\le 3x\}$

$$\iint_D xy\, dA = \boxed{\int_0^3 \int_{x^2}^{3x} xy\, dy\, dx} \quad (1)$$

$$= \int_0^3 \left[x\frac{1}{2}y^2\right]_{y=x^2}^{y=3x} dx$$

$$= \int_0^3 \frac{x}{2}\left[(3x)^2-(x^2)^2\right] dx$$

(2)

$$= \int_0^3 \frac{x}{2}(9x^2-x^4)\, dx = \frac{1}{2}\int_0^3 (9x^3-x^5)\, dx$$

$$= \frac{1}{2}\left[9\frac{x^4}{4}-\frac{x^6}{6}\right]_0^3 = \frac{1}{2}\left[\frac{9}{4}\cdot 3^4-\frac{3^6}{6}\right]$$

$$= \frac{3^4}{2}\left(\frac{9}{4}-\frac{3^2}{6}\right) = \frac{81}{2}\times 9\times\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{729}{2}\cdot\frac{1}{12}$$

$$= \frac{729}{24} = \boxed{\frac{243}{8}} \quad (1)$$

- Type II: $D = \{(x,y) \mid 0 \le y \le 9, \ \frac{y}{3} \le x \le \sqrt{y}\}$

$$\iint_D xy \, dA = \boxed{\int_0^9 \int_{\frac{y}{3}}^{\sqrt{y}} xy \, dx \, dy} \quad 1$$

$$= \int_0^9 \left[y \frac{x^2}{2} \right]_{x=\frac{y}{3}}^{x=\sqrt{y}} dy = \int_0^9 \frac{y}{2}\left[\sqrt{y}^2 - \left(\frac{y}{3}\right)^2 \right] dy$$

2

$$= \int_0^9 \frac{y}{2}\left(y - \frac{y^2}{9}\right) dy = \frac{1}{2}\int_0^9 \left(y^2 - \frac{y^3}{9}\right) dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{1}{9}\frac{y^4}{4} \right]_0^9 = \frac{1}{2}\left(\frac{9^3}{3} - \frac{9^4}{36} \right)$$

$$= \frac{9^3}{2}\left(\frac{1}{3} - \frac{9}{36} \right) = \frac{81 \times 9}{2}\left(\frac{1}{3} - \frac{1}{4} \right)$$

$$= \frac{81 \times 9}{2} \frac{1}{12} = \frac{81 \times 3}{2 \times 4} = \boxed{\frac{243}{8}} \quad 1$$

(the same as Type I)

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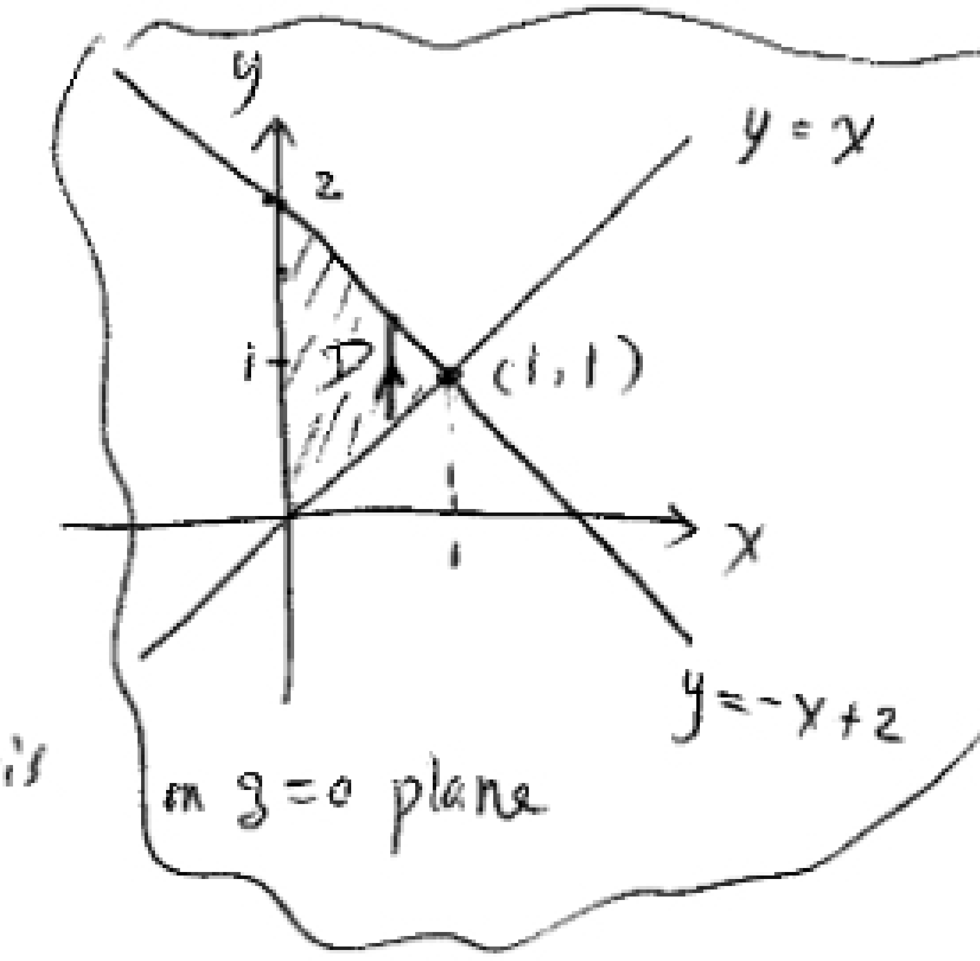
- on $z=0$ plane (i.e. xy-plane): the $y=x$ and $x+y=2$ planes intersect $z=0$ plane as lines $y=x$ and $x+y=2$.



- The $z=x$ plane through the y-axis

So the volume of the solid is

$$V = \iint_D x \, dA, \text{ where } D = \{(x,y) \mid 0 \le x \le 1, \ x \le y \le -x+2\}$$

$$2 \ \boxed{V = \int_0^1 \int_x^{-x+2} x \, dy \, dx} = \int_0^1 [xy]_{y=x}^{y=-x+2} dx = \int_0^1 x(-x+2-x) \, dx \quad 2$$

$$= \int_0^1 x(2-2x) \, dx = \int_0^1 (2x - 2x^2) \, dx = \left[x^2 - \frac{2}{3}x^3 \right]_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}} \quad 1$$

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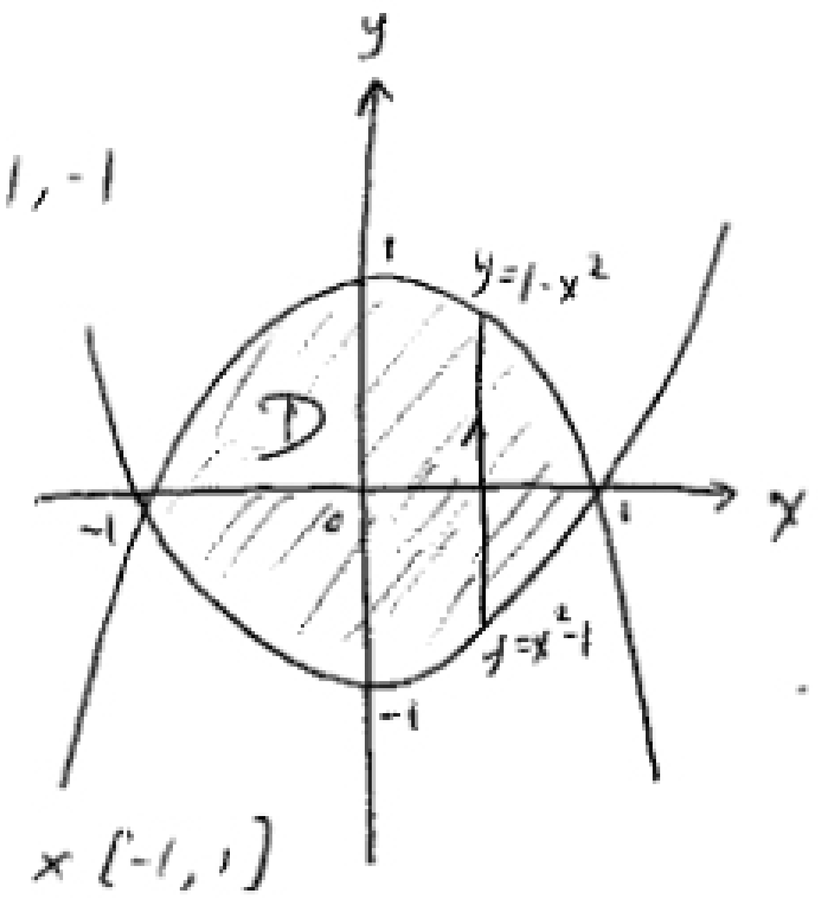
$$\begin{cases} y = 1 - x^2 \\ y = x^2 - 1 \end{cases} \Rightarrow 1 - x^2 = x^2 - 1 \Rightarrow x^2 = 1 \Rightarrow x = 1, -1$$



The region D:

$$D = \{ (x, y) \mid -1 \le x \le 1,\ x^2 - 1 \le y \le 1 - x^2 \}$$

The region D is inside $[-1,1] \times [-1,1]$

Plane 1: $z = 2 - (x + y)$

Plane 2: $z = 2(x+y) + 10$

In $[-1,1]$: $\left[2(x+y)+10\right] - \left[2 - (x+y)\right]$

$$= 3(x+y) + 8 \geqslant 3[-1+(-1)] + 8 = 2$$

So plane 2 is above plane 1 on D.

The volume:

$$V = \iint_D \left[3(x+y) + 8\right] dA = \int_{-1}^{1} \int_{x^2-1}^{1-x^2} (3x + 3y + 8)\, dy\, dx$$

$$= \int_{-1}^{1} \left[3xy + 3\frac{y^2}{2} + 8y\right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \left\{3x\left[1 - x^2 - (x^2-1)\right] + \frac{3}{2}\left[(1-x^2)^2 - (x^2-1)^2\right] + 8\left[(1-x^2) - (x^2-1)\right]\right\} dx$$

$$= \int_{-1}^{1} \left[3x(2 - 2x^2) + 8(2 - 2x^2)\right] dx = \int_{-1}^{1} (6x - 6x^3 + 16 - 16x^2)\, dx$$

$$= \int_{-1}^{1} (16 - 16x^2)\, dx = \left[16x - 16\frac{x^3}{3}\right]_{-1}^{1} = 16 \times 2 - \frac{16}{3} \times 2 = 16\left(2 - \frac{2}{3}\right)$$

$$= \frac{64}{3}$$