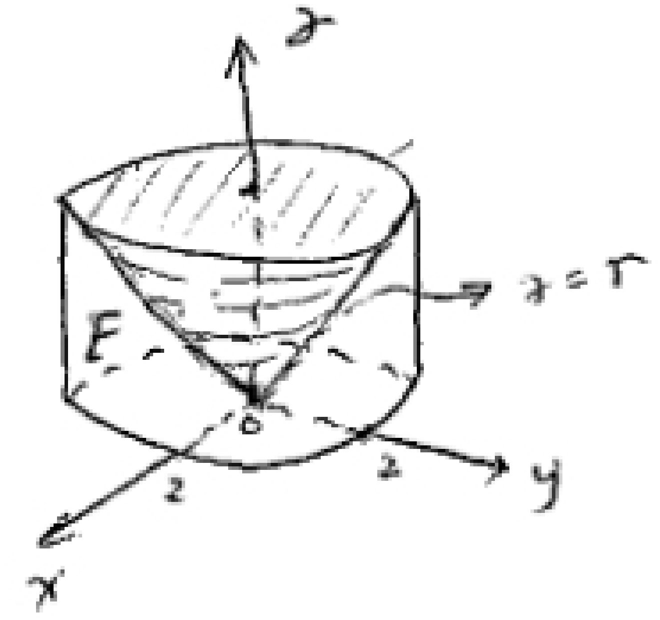
MATH2339, HW35: Solution (16)

§12.6

4 16/.



$E = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le r\}$

The solid is the white part in the figure (the unshaded part).

3

$$I = \int_0^2 \int_0^{2\pi} \int_0^r r \, dz \, d\theta \, dr$$

$$= \int_0^2 \int_0^{2\pi} \big[rz \big]_{z=0}^{z=r} \, d\theta \, dr = \int_0^2 \int_0^{2\pi} r^2 \, d\theta \, dr$$

$$= \int_0^2 \big[r^2 \theta \big]_{\theta=0}^{2\pi} \, dr = \int_0^2 2\pi r^2 \, dr = 2\pi \left[\frac{r^3}{3} \right]_0^2 = \boxed{\frac{16\pi}{3}}.$$

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4 18/.

Intersection: $\begin{cases} z = x^2 + y^2 = r^2 \\ z = 4 \end{cases} \Rightarrow \begin{cases} r = 2 \\ z = 4 \end{cases}$

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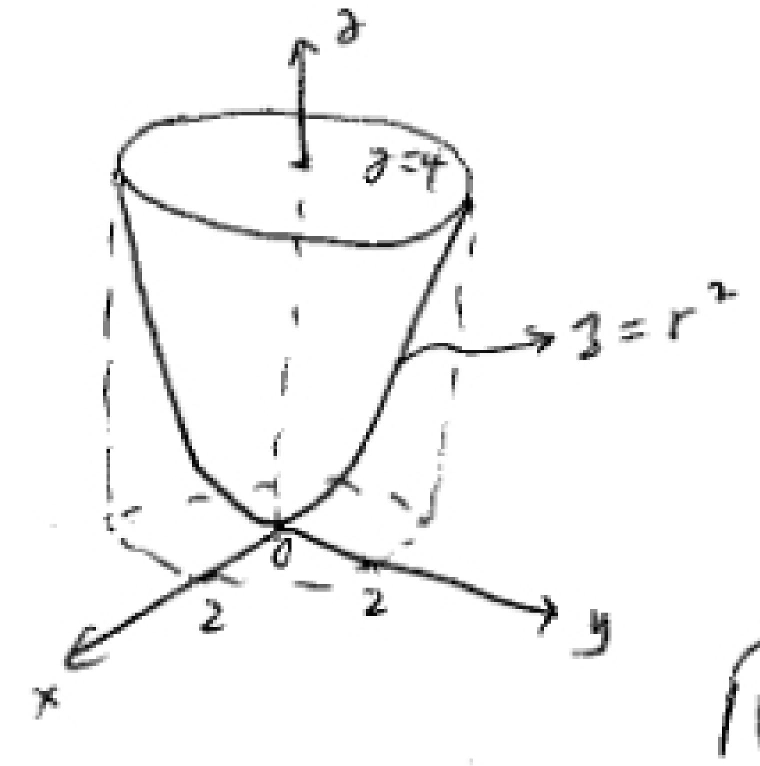
In cylindrical coordinates:

$$E = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r^2 \le z \le 4\}$$

So: $$\iiint_E z \, dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 (z) r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[\frac{1}{2} z^2 r \right]_{z=r^2}^{z=4} dr \, d\theta$$



$$= \int_0^{2\pi} \int_0^2 \frac{1}{2} r \, (16 - r^4) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \frac{1}{2} (16 r - r^5) \, dr \, d\theta = \left(\int_0^{2\pi} d\theta \right) \frac{1}{2} \left[16 \frac{r^2}{2} - \frac{r^6}{6} \right]_{r=0}^{r=2}$$

$$= 2\pi \cdot \frac{1}{2} \cdot \left(8 \cdot 2^2 - \frac{2^6}{6} \right) = \pi \left(32 - \frac{32}{3} \right) = \boxed{\frac{64}{3} \pi}$$ 1

4 20/. $I = \iiint_E x \, dV$

In cylindrical coordinates: $E = \{ (r, \theta, z) \mid 0 \le r \le 2, \ 0 \le \theta \le \pi, \ 0 \le z \le r\cos\theta + r\sin\theta + 5 \}$

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So $$I = \int_0^{2\pi} \int_2^3 \int_0^{r\cos\theta + r\sin\theta + 5} (r \cos\theta) \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_2^3 \left[r^2 \cos\theta \, z \right]_{z=0}^{z = r\cos\theta + r\sin\theta + 5} dr \, d\theta$$

$$= \int_0^{2\pi} \int_2^3 r^2 \cos\theta \, (r\cos\theta + r\sin\theta + 5) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_2^3 (\cos^2\theta \, r^3 + \sin\theta \cos\theta \, r^3 + 5 \cos\theta \, r^2) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\cos^2\theta \frac{r^4}{4} + \sin\theta \cos\theta \frac{r^4}{4} + 5 \cos\theta \frac{r^3}{3} \right]_{r=2}^{r=3} d\theta$$

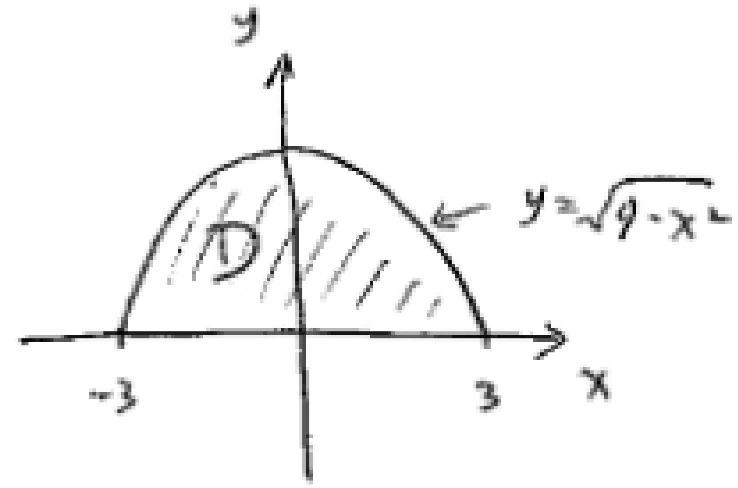
$$= \int_0^{2\pi} \left[\frac{81 - 16}{4} \cos^2\theta + \frac{81 - 16}{4} \sin\theta \cos\theta + 5 \left(\frac{27 - 8}{3} \cos\theta \right) \right] d\theta$$

$$= \frac{65}{4} \int_0^{2\pi} \cos^2\theta \, d\theta + \frac{65}{4} \int_0^{2\pi} \sin\theta \cos\theta \, d\theta + \frac{5 \times 19}{3} \int_0^{2\pi} \cos\theta \, d\theta$$

$$= \frac{65}{4} \int_0^{2\pi} \frac{\cos 2\theta + 1}{2} \, d\theta + \frac{65}{4} \int_0^{2\pi} \sin\theta \, d(\sin\theta) + \frac{5 \times 19}{3} \left[\sin\theta \right]_0^{2\pi}$$

$$= \frac{65}{8} \left[\frac{\sin 2\theta}{2} + \theta \right]_0^{2\pi} + \frac{65}{4} \left[\frac{\sin^2\theta}{2} \right]_0^{2\pi} = \frac{65}{8} \times 2\pi = \boxed{\frac{65\pi}{4}}$$ 1

30/ $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$



$$\left.\begin{array}{l} x: -3 \to 3 \\ y: 0 \to \sqrt{9-x^2} \end{array}\right\} \Rightarrow D$$

$$D = \{ (r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le \pi \}$$

$z: 0 \to 9-x^2-y^2$, i.e. $z: 0 \to 9-r^2$

So: $$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$

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$$= \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r \; r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{3} \left[r^2 z \right]_{z=0}^{z=9-r^2} dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{3} r^2(9-r^2)\,dr\,d\theta = \int_0^{\pi}\left[9\frac{r^3}{3} - \frac{r^5}{5} \right]_{r=0}^{r=3} d\theta$$

$$= \pi\cdot\left(3\cdot 3^3 - \frac{3^5}{5}\right) = \pi\left(81 - \frac{81\times 3}{5}\right) = 81\pi\left(1-\frac{3}{5}\right)$$

$$= \frac{81\pi\times 2}{5} = \boxed{\frac{162\pi}{5}}$$

1