Math 2339 , HW37 : Solution 10

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$H = \{ (x, y, z) \mid x^2 + y^2 + z^2 \le 9, \; z \ge 0 \}$

0.5 $\boxed{= \{ (\rho, \theta, \phi) \mid 0 \le \rho \le 3, \; 0 \le \theta \le 2\pi, \; 0 \le \phi \le \frac{\pi}{2} \}}$

$$\iiint_H (9 - x^2 - y^2)\, dV = \boxed{\int_0^{\pi/2} \int_0^{2\pi} \int_0^3 \left[9 - (\rho \sin\phi \cos\theta)^2 - (\rho \sin\phi \sin\theta)^2 \right] \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}$$

1 1

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$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 \left[9 - \rho^2 \sin^2\phi \, (\cos^2\theta + \sin^2\theta) \right] \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^3 (9 - \rho^2 \sin^2\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \int_0^3 (9\rho^2 \sin\phi - \rho^4 \sin^3\phi)\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \left[9\sin\phi \frac{\rho^3}{3} - \sin^3\phi \frac{\rho^5}{5} \right]_{\rho=0}^{\rho=3} d\theta\, d\phi$$

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$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \left(81 \sin\phi - \frac{243}{5} \sin^3\phi \right) d\phi$$

$$= 2\pi \left(\left[-81 \cos\phi \right]_0^{\pi/2} - \frac{243}{5} \int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \right)$$

$$= 2\pi \left[-81 (0 - 1) - \frac{243}{5} \int_0^{\frac{\pi}{2}} \sin\phi (1 - \cos^2\phi)\, d\phi \right]$$

$$= 2\pi \left[81 - \frac{243}{5} \left(\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi + \int_0^{\frac{\pi}{2}} \cos^2\phi \cdot d(\cos\phi) \right) \right]$$

$$= 2\pi \left[81 - \frac{243}{5} \left(\left[-\cos\phi \right]_0^{\pi/2} + \left[\frac{1}{3} \cos^3\phi \right]_0^{\frac{\pi}{2}} \right) \right]$$

$$= 2\pi \left[81 - \frac{243}{5} \left(1 + \frac{1}{3}(0 - 1) \right) \right] = 2\pi \left(81 - \frac{243}{5} \times \frac{2}{3} \right)$$

$$= 2\pi \times 81 \left(1 - \frac{2}{5} \right) = 2\pi \times 81 \times \frac{3}{5} = \boxed{\frac{486}{5}\pi}$$ 0.5

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28/ In spherical coordinates:

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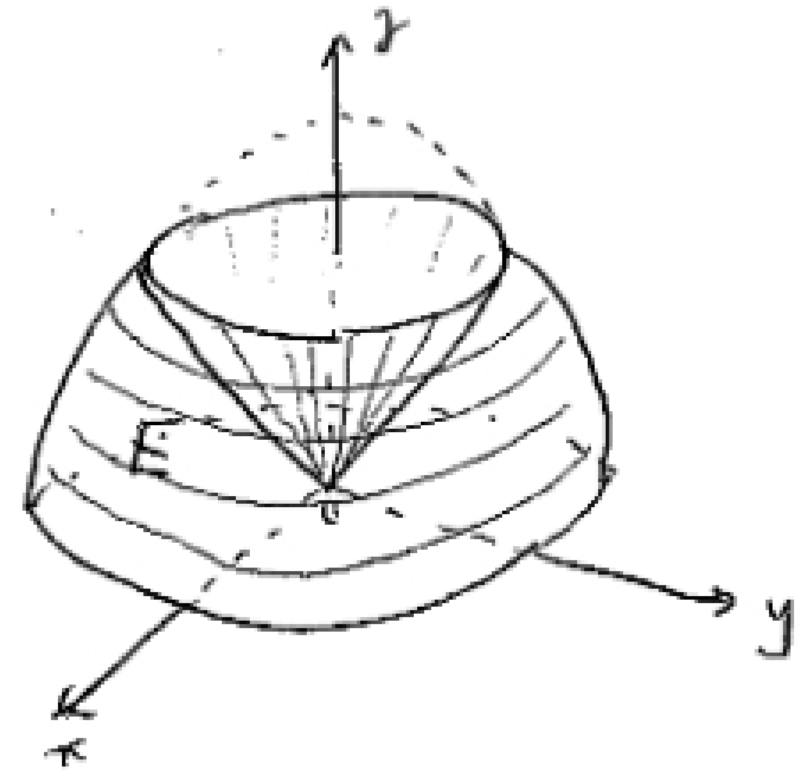
sphere $x^2+y^2+z^2=4$: $\rho = 2$

cone $z=\sqrt{x^2+y^2}$: $\rho\cos\phi = \rho\sin\phi \Rightarrow \phi = \frac{\pi}{4}$

the solid: $\boxed{E = \{(\rho,\theta,\phi) \mid 0\le\rho\le 2,\ 0\le\theta\le 2\pi,\ \frac{\pi}{4}\le\phi\le\frac{\pi}{2}\}}$

0.5



The volume of the solid E:

$$\boxed{V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 1\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$ 1

1

$$= \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\left[\frac{1}{3}\rho^3\right]_{\rho=0}^{\rho=2}\sin\phi\,d\theta\,d\phi$$

$$= \frac{8}{3}\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\sin\phi\,d\theta\,d\phi = \frac{16\pi}{3}\int_{\pi/4}^{\pi/2}\sin\phi\,d\phi$$

$$= \frac{16}{3}\left[-\cos\phi\right]_{\pi/4}^{\pi/2} = \frac{16}{3}\cdot\frac{\sqrt{2}}{2} = \boxed{\frac{8\sqrt{2}}{3}\pi}$$ 0.5

40. The solid region modeling the earth's atmosphere is
(3)
a spherical shell (from the ground to an altitude of 5000 m, i.e. $6.370 \times 10^6 \le \rho \le 6.375 \times 10^6$). It is given by

$$\boxed{E = \{(\rho, \theta, \phi) \mid 6.370 \times 10^6 \le \rho \le 6.375 \times 10^6,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}}$$

The mass of the atmosphere in this region is (0.5)

$$\boxed{M = \iiint_E \delta \, dV = \int_0^{\pi} \int_0^{2\pi} \int_{6.370 \times 10^6}^{6.375 \times 10^6} (619.09 - 0.000097\rho)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi}$$ (1)

(1.5)

$$= \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_{6.370 \times 10^6}^{6.375 \times 10^6} (619.09 - 0.000097\rho)\rho^2 \, d\rho$$

$$= \left[-\cos\phi\right]_0^{\pi} \cdot 2\pi \left[\frac{1}{3} 619.09\rho^3 - \frac{1}{4} 0.000097\rho^4\right]_{6.370 \times 10^6}^{6.375 \times 10^6}$$

$$\approx 2 \times 2\pi \times (1.944 \times 10^{17}) \approx 2.44 \times 10^{18} \ [kg].$$