
MATH 2300 - Calculus III

Fall 2008

Homework 4 - Solutions

1. Use Lagrange Multipliers to find the minimum value of $f(x, y, z) = 6x - y^2 + xz + 60$ subject to the constraint $x^2 + y^2 + z^2 = 36$.

SOLUTION:

We wish to optimize $f(x, y, z) = 6x - y^2 + xz + 60$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 36$. First, we must set up and solve:

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ (6+z)\mathbf{i} + (-2y)\mathbf{j} + x\mathbf{k} &= \lambda[2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}]\end{aligned}$$

Setting the like components equal, we have:

$$6+z = 2\lambda x, \quad -2y = 2\lambda y, \quad x = 2\lambda z$$

From the second equation, we have that:

$$2y + 2\lambda y = 0 \implies 2y(\lambda + 1) = 0 \implies y = 0 \text{ or } \lambda = -1$$

Taking this in cases, if $y = 0$, we have the remaining equations (including the constraint):

$$6+z = 2\lambda x, \quad x = 2\lambda z, \quad g(x, 0, z) = x^2 + z^2 = 36$$

Solving for λ , we have:

$$\lambda = \frac{6+z}{2x} = \frac{x}{2z} \text{ for } x, z \neq 0 \implies 2x^2 = 12z + 2z^2 \implies x^2 = 6z + z^2$$

Substituting this into the constraint gives us:

$$6z + z^2 + z^2 = 36 \implies z^2 + 3z - 18 = 0 \implies (z+6)(z-3) = 0$$

For the value $z = -6$, we have $x^2 = 6(-6) + (-6)^2 = 0$, and the point $(0, 0, -6)$.

For the value $z = 3$, we have $x^2 = 6(3) + (3)^2 = 27$, and the points $(\pm 3\sqrt{3}, 0, 3)$.

Taking the second case, if $\lambda = -1$, we have the remaining equations:

$$6+z = -2x, \quad x = -2z, \quad x^2 + y^2 + z^2 = 36$$

$$6+z = -2(-2z) \implies z = 2, \quad x = -4$$

For this case, we have $(-4)^2 + y^2 + 2^2 = 36$, $\implies y = \pm 4$, and the points $(-4, \pm 4, 2)$.

Finally, testing the function f at each point gives us:

$$f(0, 0, -6) = 60$$

$$f(3\sqrt{3}, 0, 3) = 60 + 27\sqrt{3} \approx 106.8$$

$$f(3\sqrt{3}, 0, 3) = 60 - 27\sqrt{3} \approx 13.23$$

$$f(-4, \pm 4, 2) = 12$$

Therefore, the minimum value is 12, which occurs at the two points $(-4, \pm 4, 2)$.

2. Find the radius of the base and the height of the right circular cylinder of largest volume that can be inscribed in a sphere of radius a .

SOLUTION:

Let the radius of the base of the cylinder be r and its height be h . Also, let's imagine the central axis of the cylinder is on the z -axis. By drawing a picture and using the equation of the sphere ($x^2 + y^2 + z^2 = 1$), it is easy to see that we must have $h^2/4 + r^2 = a^2$ (for example, in the yz -plane the top of the cylinder intersects the sphere at the point $(x, y, z) = (0, r, h/2)$). The volume of the cylinder is $V = \pi r^2 h$. After substitution, we can write the volume as $V = \pi h(a^2 - h^2/4)$. We have that $dV/dh = \pi(a^2 - 3h^2/4)$. The critical point is $h = 2a/\sqrt{3}$. It follows that the dimensions of the cylinder with maximum volume are $h = 2a/\sqrt{3}$ and $r = a\sqrt{2/3}$.

3. Evaluate $\int_1^2 \int_0^{\pi/x} x^2 \sin(xy) dy dx$.

SOLUTION:

$$\int_1^2 \int_0^{\pi/x} x^2 \sin(xy) dy dx = \int_1^2 x^2 \left(\frac{-\cos(xy)}{x} \right) \Big|_0^{\pi/x} dx = \int_1^2 -x[\cos(\pi) - \cos(0)] dx = \int_1^2 (2x) dx = x^2 \Big|_1^2 = 2^2 - 1^2 = 3$$

4. Find the integral of the function $f(x, y, z) = xyz$ over the set $\{(x, y, z) : x \geq 0, y \geq 0, x + y \leq 1, 0 \leq z \leq 1\}$.

SOLUTION:

The region is cylinder over a triangular base. One way to set up the integration is the triple integral

$$\int_0^1 \int_0^1 \int_0^{1-y} xyz \, dx dy dz = \frac{1}{48}.$$

5. Evaluate $\int \int_R y \, dA$, where R is the region in the xy -plane bounded by $x = y^2$ and $x = 2y - y^2$.

SOLUTION:

The points of intersection of the two curves are obtained by solving the equations simultaneously as follows:

$$x = y^2 = 2y - y^2 \implies 2y^2 - 2y = 2y(y - 1) = 0$$

Thus, the points of intersection are $(0, 0)$, and $(1, 1)$. We also notice (perhaps by drawing a picture) that on the region of interest $y^2 < 2y - y^2$. Using these facts the setup and evaluation of the integral are easy:

$$\begin{aligned} \int \int_R y \, dA &= \int_0^1 \int_{y^2}^{2y-y^2} y \, dx dy = \int_0^1 xy \Big|_{y^2}^{2y-y^2} dy = \int_0^1 y(2y - y^2 - y^2) dy \\ &= \int_0^1 (2y^2 - 2y^3) dy = \frac{2}{3}y^3 - \frac{1}{2}y^4 \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \end{aligned}$$

6. Evaluate the integral $\int_0^1 \int_0^1 x^3 e^{-x^2 y} \, dx dy$.

SOLUTION:

The easy way to evaluate the integral is to interchange the order of integration and proceed as follows:

$$\begin{aligned} \int_0^1 \left(\int_0^1 x^3 e^{-x^2 y} dy \right) dx &= \int_0^1 -x e^{-x^2} + x dx \\ &= \frac{1}{2} e^{-x^2} + \frac{1}{2} x \Big|_0^1 \\ &= \frac{1}{2e}. \end{aligned}$$

7. Use polar coordinates to find the volume of the solid bounded below by the paraboloid $z = 3x^2 + 3y^2$ and above by the cone $z = 6\sqrt{x^2 + y^2}$.

SOLUTION:

The projection R of the solid region between the cone and the paraboloid onto the xy -plane is a disk. To find the radius of the bounding disk, we find the intersection of the two surfaces.

$$z = 3x^2 + 3y^2 = 6\sqrt{x^2 + y^2} \implies 3r^2 = 6r \implies r = 0, 2$$

Therefore, we can evaluate a double integral in polar coordinates to find the volume using:

$$\iint_R (z_{\text{top}} - z_{\text{bot}}) dA = \int_0^{2\pi} \int_0^2 (6r - 3r^2)r dr d\theta = \int_0^{2\pi} \left(2r^3 - \frac{3}{4}r^4 \right) \Big|_0^2 d\theta = \int_0^{2\pi} 4 d\theta = 8\pi$$

8. Find the mass of the lamina bounded by the curves $y = \sqrt{9 - x^2}$, $y = x$ and $x = 0$, in the first quadrant, if the density is given by $\rho(x, y) = \frac{3x}{y^2}$.

SOLUTION:

The mass of the lamina is found through the double integral below, which we set up in polar coordinates:

$$\begin{aligned} \text{mass} &= \iint_R \rho(x, y) dA = \int_{\pi/4}^{\pi/2} \int_0^3 \frac{3r \cos \theta}{r^2 \sin^2 \theta} r dr d\theta = \int_{\pi/4}^{\pi/2} \int_0^3 \frac{3 \cos \theta}{\sin^2 \theta} dr d\theta = \int_{\pi/4}^{\pi/2} \frac{3 \cos \theta}{\sin^2 \theta} r \Big|_0^3 d\theta \\ &= \int_{\pi/4}^{\pi/2} \frac{9 \cos \theta}{\sin^2 \theta} d\theta = -\frac{9}{\sin \theta} \Big|_{\pi/4}^{\pi/2} = -\frac{9}{\sin(\pi/2)} + \frac{9}{\sin(\pi/4)} = -9 + \frac{9}{\sqrt{2}/2} = \frac{18 - 9\sqrt{2}}{\sqrt{2}} \approx 3.73 \end{aligned}$$

9. Determine the region R in space such that the triple integral

$$\iiint_R (1 - 10x^2 - 5y^2 - z^2) dV$$

takes on its maximum value. Hint: This is a thought question. No computation is necessary.

SOLUTION: The integrand is positive on the region bounded by the ellipsoid $10x^2 - 5y^2 - z^2 = 1$, zero on the ellipsoid, and negative outside this ellipsoid. So, the maximum possible value of the integral occurs for R equal to the region bounded by this ellipsoid.

10. Set up the integral $\iiint_D f(x, y, z) dV$ where D is the first-octant portion of the solid bounded by the cone $x^2 = y^2 + z^2$ ($x \geq 0$), and the plane $x = 2$.