Homework assignment #5 solutions, MATH309

Problem 1

(a) $\begin{vmatrix} 3 & 1 & 4 \\ 2 & 3 & 5 \\ 1 & 2 & 2 \end{vmatrix} \xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - 3R_3} \begin{vmatrix} 0 & -5 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} -5 & -2 \\ -1 & 1 \end{vmatrix} = -5 - 2 = -7 \neq 0$

$\Downarrow$

$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$ is a basis in R^3

(b) Since # vector > dim (i.e. $4 > 3$) these vectors are $\boxed{\text{linearly dependent}}$

To choose a basis of their span transform to row echelon form:

$\begin{pmatrix} 3 & 1 & 6 & 1 \\ 2 & 3 & 11 & 10 \\ 1 & 2 & 7 & 3 \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\sim} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 2 & 3 & 11 & 10 \\ 3 & 1 & 6 & 1 \end{pmatrix} \underset{R_3 \to R_3 - 3R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 0 & -1 & -3 & 4 \\ 0 & -5 & -15 & -8 \end{pmatrix} \sim$

$\overset{R_2 \to -R_2}{\sim} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 5 & 15 & 8 \end{pmatrix} \overset{R_3 \to R_3 - 5R_1}{\sim} \begin{pmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 28 \end{pmatrix}$

# of leading elements $= 3 \Rightarrow$ our set of vector is a spanning set in $\mathbb{R}^3$ and also since the leading elements are in the 1st, 2nd and 4th columns the 1st, 2nd, and 4th vectors form a basis, i.e.

$\boxed{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 10 \\ 3 \end{pmatrix} \text{ form a basis of } R^3}$

$\boxed{\text{dim of span} = 3}$

1c) $\begin{vmatrix} 1 & 4 & 6 & -7 \\ 3 & 0 & 6 & 3 \\ 0 & 2 & 2 & -4 \\ 5 & 1 & 11 & 3 \end{vmatrix} \xrightarrow[R_4 \to R_4 - 5R_1]{R_2 \to R_2 - 3R_1} \begin{vmatrix} 1 & 4 & 6 & -7 \\ 0 & -12 & -12 & 24 \\ 0 & 2 & 2 & -4 \\ 0 & -19 & -19 & 38 \end{vmatrix} =$

$= -12 \cdot 2 \cdot (-19) \begin{vmatrix} 1 & 4 & 6 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow$ because the matrix contains equal rows

the vector are linearly dependent

To sort out the basis continue the same calculation to transform the corresponding matrix to the row echelon form (in the calculation of the determinant we used the same row operation as we use for such transformation to row echelon form)

$\begin{pmatrix} 1 & 4 & 6 & -7 \\ 3 & 0 & 6 & 3 \\ 0 & 2 & 2 & -4 \\ 5 & 1 & 11 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 6 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \underset{R_4 \to R_4 - R_2}{\overset{R_3 \to R_3 - R_2}{\sim}}$

$\sim \begin{pmatrix} 1 & 4 & 6 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The leading elements correspond to the first and the second vectors $\Rightarrow$ the first and the second vectors form a basis of the span of our four vectors.

i.e $\boxed{\begin{pmatrix}1\\3\\0\\5\end{pmatrix} \text{ and } \begin{pmatrix}4\\0\\2\\1\end{pmatrix}}$ form a basis

of the span and dim of span is $\boxed{2}$

d) The problem is equivalent to the same question for the vectors

$$\begin{pmatrix}3\\-2\\4\\1\end{pmatrix}, \begin{pmatrix}-1\\3\\2\\5\end{pmatrix}, \begin{pmatrix}11\\-12\\8\\-7\end{pmatrix}, \begin{pmatrix}2\\1\\6\\4\end{pmatrix} \text{ in } \mathbb{R}^4$$

Transform the matrix with these columns to the row echelon form

$$\begin{pmatrix}3 & -1 & 11 & 2\\ -2 & 3 & -12 & 1\\ 4 & 2 & 8 & 6\\ 1 & 5 & -7 & 4\end{pmatrix} \underset{R_1 \leftrightarrow R_4}{\sim} \begin{pmatrix}1 & 5 & -7 & 4\\ -2 & 3 & -12 & 1\\ 4 & 2 & 8 & 6\\ 3 & -1 & 11 & 2\end{pmatrix}$$

$$\underset{\substack{R_2 \to R_2+2R_1\\ R_3 \to R_3-4R_1\\ R_4 \to R_4-3R_1}}{\sim} \begin{pmatrix}1 & 5 & -7 & 4\\ 0 & 13 & -26 & 9\\ 0 & -18 & 36 & -10\\ 0 & -16 & 32 & -10\end{pmatrix} \underset{\substack{R_2 \to \frac{1}{13}R_2\\ R_3 \to -\frac{1}{18}R_3\\ R_4 \to -\frac{1}{16}R_4}}{\sim}$$