MATH 2300 - Calculus III

Fall 2008

Homework 6 - DUE Wednesday 19 November

1. Find the work done as a particle moves along the curve $x = \cos t$, $y = \sin t$ from $(x, y) = (1, 0)$ to $(x, y) = (-1, 0)$ in the force field $\mathbf{F}$ given by
$$\mathbf{F}(x, y) = (2xe^{xy} \sin x + x^2 e^{xy} \cos x + yx^2 \sin xe^{xy})\mathbf{i} + x^3 e^{xy} \sin x\mathbf{j}.$$

2. Suppose that T is a positive number, f and g have continuous partial derivatives and C is a curve in the plane given by $r(t) = (x(t), y(t))$ for $0 \le t \le T$ such that $x'(t) = f(x(t), y(t))$ and $y'(t) = g(x(t), y(t))$.
(a) Show that
$$\int_C -g\,dx + f\,dy = 0.$$
(b) Suppose that $\mathbf{H}$ is the vector field in the plane given by $\mathbf{H}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$, C (the curve with the properties given in Part (a)) is a simple closed curve, and Ω is the plane region bounded by C. Show that
$$\int_\Omega \text{div}\, H\, dA = 0.$$
(c) For the case where $f(x, y) = x^3 - y$ and $g(x, y) = y^3 + x$, show that no such simple closed curve C cannot exist.

3. Suppose that f is a complex valued function of two variables; that is, $f(x, y) = u(x, y) + iv(x, y)$ where $i := \sqrt{-1}$. For a simple closed curve C in the plane, define
$$\int_C f\,dz = \int_C (u + iv)(dx + idy) = \int_C (u\,dx - v\,dy) + i \int_C v\,dx + u\,dy.$$
(You can think of $z = x + iy$ and $dz = dx + idy$ to see where the formula comes from.) Suppose that the two functions u and v have continuous partial derivatives and the Cauchy-Riemann relations $u_x = v_y$ and $v_x = -u_y$ hold among the partial derivatives of u and v. Prove (the special case of) Cauchy's theorem:
$$\int_C f\,dz = 0.$$

4. Use Green's theorem to find the area of the limaçon given by $\mathbf{r}(t) = \langle 2\cos t - \cos 2t, 2\sin t - \sin 2t\rangle$, $0 \le t \le 2\pi$.

5. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are differentiable vector functions of time.
(a) Show that
$$\frac{d}{dt}(\mathbf{v} \cdot \mathbf{w}) = \frac{d}{dt}\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \frac{d}{dt}\mathbf{w}.$$
Hint: Write the vector functions in components and use the product rule of differentiation.
(b) Show that
$$\frac{d}{dt}(\mathbf{v} \times \mathbf{w}) = \frac{d}{dt}\mathbf{v} \times \mathbf{w} + \mathbf{v} \times \frac{d}{dt}\mathbf{w}.$$

6. Suppose that F is a differentiable scalar function of a scalar variable and $\mathbf{r}$ is the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\nabla(F(\mathbf{r}\cdot\mathbf{r})) = F'(\mathbf{r}\cdot\mathbf{r})2\mathbf{r}$.

7. By Newton's 2nd Law, the equation of motion of a particle of mass m moving in the plane under the influence of a potential $U = U(\mathbf{r})$ (which is a scalar valued function of the vector variable $\mathbf{r}$) can be written in the form

$$m\frac{d^2\mathbf{r}}{dt^2} = -\nabla U(\mathbf{r}),$$

where the position of the particle is $\mathbf{r}$.
(a) The total energy of the particle (the sum of its kinetic and potential energies) is $E = \frac{m}{2}(\mathbf{v}\cdot\mathbf{v}) + U(\mathbf{r})$ where $\mathbf{v}$ is the velocity of the particle. Show that $dE/dt = 0$; that is, energy is conserved as the particle moves.
(b) The angular momentum of the particle is $\mathbf{A} = \mathbf{r}\times m\mathbf{v}$. Suppose the potential U is circularly symmetric; that is, $U(\mathbf{r}) = F(\mathbf{r}\cdot\mathbf{r})$, where F is some scalar function of a scalar variable. Show that the angular momentum is conserved.
(c) The gravitational potential is $GM/|\mathbf{r}|$. Is the angular momentum conserved for a particle moving in a gravitational field? (Explain.)

8. Recall Laplace's equation $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$. For which value(s) of α does the function u given by

$$u(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^\alpha}$$

satisfy Laplace's equation?

9. Compute the line integral of the vector field $\mathbf{F}$ given by

$$\mathbf{F} = e^{-(xy)^2}(4x^3 - 2(x^4 + y^4)xy^2)\mathbf{i} + e^{-(xy)^2}(4y^3 - 2(x^4 + y^4)x^2y)\mathbf{i}$$

around the curve C given by $x(t) = 3\cos t$ and $y(t) = 2\sin t$ for $0 \le t \le 2\pi$.

10. (a) Suppose that (x_1, y_1) and (x_2, y_2) are points in the plane and C is the line segment from (x_1, y_1) to (x_2, y_2). Show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1.$$

(b) Suppose that (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) are the vertices of a polygon given in counterclockwise order. Show that the area of the polygon is

$$\frac{1}{2}((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)).$$

(c) Check that the formula is true for all rectangles.