Homework assignment #8 MATH 309 Solution

Problem 1

a) $A = \begin{pmatrix} -7 & 6 \\ -15 & 12 \end{pmatrix}$

i) eigenvalues:

Char equation: $\det(A-\lambda I)=0 \overset{n=2}{\Leftrightarrow}$

$$\lambda^2 - \operatorname{tr}A\,\lambda + \det(A) = 0$$

$\operatorname{tr}A = -7+12 = 5$

$\det(A) = -84+90 = 6 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$

$D = 25-24 = 1$ or $\lambda^2-5\lambda+6 = (\lambda-2)(\lambda-3) = 0$

$\lambda_1 = \frac{5+1}{2} = 3$

$\lambda_2 = \frac{5-1}{2} = 2$

$\lambda_1 = 3,\ \lambda_2 = 2$

ii) eigenspaces:

$\underline{\lambda_1 = 3}$

$(A-3I)v = 0 \Leftrightarrow \begin{pmatrix} -10 & 6 \\ -15 & 9 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -10 & 6 \\ -15 & 9 \end{pmatrix} \underset{R_2 \to \frac{1}{3}R_2}{\overset{R_1 \to \frac{1}{2}R_1}{\sim}} \begin{pmatrix} -3 & 5 \\ -3 & 5 \end{pmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} -3 & 5 \\ 0 & 0 \end{pmatrix} \Rightarrow -5v_1 + 3v_2 = 0 \Rightarrow v_1 = \frac{3}{5}v_2 \Rightarrow E_3 = \operatorname{Span}\begin{pmatrix} \frac{3}{5} \\ 1 \end{pmatrix} =$

$= \operatorname{Span}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

is $\boxed{\operatorname{Span}\begin{pmatrix} 3 \\ 5 \end{pmatrix}}$

$\lambda_2 = 2$

$(A-2I)v = 0 \Leftrightarrow \begin{pmatrix} -9 & 6 \\ -15 & 10 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -9 & 6 \\ -15 & 10 \end{pmatrix} \underset{R_2 \to \frac{1}{5}R_2}{\overset{R_1 \to \frac{1}{3}R_1}{\sim}} \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \overset{R_2 \to R_2 - R_1}{\sim} \begin{pmatrix} -3 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow -3v_1 + 2v_2 \Rightarrow E_2 = \operatorname{Span}\begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix} =$

$v_1 = \frac{2}{3}v_2$

$= \boxed{\operatorname{Span}\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$

iii) A is diagonizable because it has distinct real roots

(iv) $\boxed{D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad X = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}}$

$A = XDX^{-1}$

(v) $A^3 \; XD^3X^{-1}$

(just $A^3 \; XD\underbrace{X^{-1}XD}_{I}\underbrace{X^{-1}X}_{I}DX^{-1} = XD^3X^{-1}$ [illegible])

$$D^4 = \begin{pmatrix} 3^3 & 0 \\ 0 & 2^3 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & 8 \end{pmatrix} \; ; \Rightarrow X^{-1} = \frac{1}{9-10}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 27 & 0 \\ 0 & 8 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 81 & 16 \\ 135 & 24 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} -243 + 80 & 162 - 48 \\ -405 + 120 & 270 - 48 \end{pmatrix} = \begin{pmatrix} -163 & 114 \\ -285 & 222 \end{pmatrix}$$

(vi) $e^{At} = X e^{Dt} X^{-1}$

$$e^{Dt} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix} \Rightarrow$$

$$e^{At} = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 3e^{3t} & 2e^{2t} \\ 5e^{3t} & 3e^{2t} \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} -9e^{3t} + 10e^{2t} & 6e^{3t} - 6e^{2t} \\ -15e^{3t} + 15e^{2t} & 10e^{3t} - 9e^{2t} \end{pmatrix}$$

vii) $x(t) = e^{At}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-9e^{3t}+10e^{2t}-6e^{3t}+6e^{2t}\\-15e^{3t}+15e^{2t}-10e^{3t}+9e^{2t}\end{pmatrix} =$

$$= \boxed{\begin{pmatrix}-15e^{3t}+16e^{2t}\\-25e^{3t}+24e^{2t}\end{pmatrix}}$$

b) $A = \begin{pmatrix}-13 & 40 & -10\\-9 & 27 & -7\\-17 & 50 & -14\end{pmatrix}$

(i) If $\lambda_1 = -3$ and $\lambda_2 = 1$ are eigenvalues of A and λ_3 is one more eigenvalue, then

$$\lambda_1+\lambda_2+\lambda_3 = tr(A) = -13+27-14 = 0 \Rightarrow$$

$$-3+1+\lambda_3 = 0 \Rightarrow \lambda_3 = 2$$

So the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 1$, $\lambda_3 = 2$

(ii) Eigenspaces

$\lambda_1 = -3$: $(A+3I)v = 0$

$$\begin{pmatrix}-10 & 40 & -10\\-9 & 30 & -7\\-17 & 50 & -11\end{pmatrix} \underset{R_1 \to \frac{1}{10}R_1}{\sim} \begin{pmatrix}-1 & 4 & -1\\-9 & 30 & -7\\-17 & 50 & -11\end{pmatrix} \xrightarrow[R_3 \to R_3 - 17R_1]{R_2 \to R_2 - 9R_1}$$

$$\begin{pmatrix}1 & 4 & -1\\0 & -6 & 2\\0 & -18 & 6\end{pmatrix} \underset{R_3 \to R_3 - R_2}{\sim} \begin{pmatrix}-1 & 4 & -1\\0 & -6 & 2\\0 & 0 & 0\end{pmatrix}$$

$-v_1 + 4v_2 - v_3 = 0$

$\Rightarrow -6v_2 + 2v_3 = 0 \Rightarrow$

$v_3 = 3v_2$ &

$-v_1 + 4v_2 - 3v_2 = -v_1 + v_2$