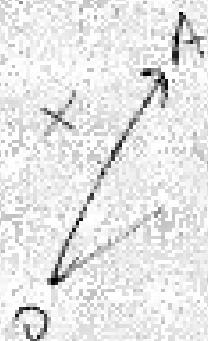


Problem 1 By Cauchy-Schwarz-Bunyakovski

$$|x^T y| \leq \|x\| \|y\| = 3 \cdot 4 = 12 \Rightarrow$$

$$\boxed{|x^T y| \leq 12} \Leftrightarrow -12 \leq x^T y \leq 12$$

Problem 2 The required point is defined by the vector projection  $p$  of the vector  $x = (1, 2, -3, 4)^T$  onto the vector  $y = (-2, 3, -4, -2)^T$



(this point is the endpoint of this vector projection, if we take the origin as the starting point)

$$p = \frac{\langle x, y \rangle}{\|y\|^2} y = \frac{-2 - 6 + 12 - 8}{4 + 9 + 16 + 4} (-2, 3, -4, -2)^T =$$

$$= \frac{-4}{33} (-2, 3, -4, -2)^T = \boxed{\left( \frac{8}{33}, -\frac{12}{33}, \frac{16}{33}, \frac{8}{33} \right)^T}$$

The distance from A to  $L = \|x - p\| = \left\| \left( 1 - \frac{8}{33}, -2 + \frac{12}{33}, -3 - \frac{16}{33}, 4 - \frac{8}{33} \right)^T \right\| =$

$$= \left\| \left( \frac{25}{33}, -\frac{54}{33}, -\frac{115}{33}, \frac{124}{33} \right)^T \right\| = \frac{1}{33} \sqrt{(25)^2 + (54)^2 + (115)^2 + (124)^2} =$$

$$= \boxed{\frac{1}{33} \sqrt{32142}}$$

Problem 3

$$\text{distance} = \frac{|2(3-2) + 3(-1+1) + 4(4-5) + 5(-2+4)|}{\sqrt{2^2 + 3^2 + 4^2 + 5^2}}$$

$$= \frac{|2 - 4 + 10|}{\sqrt{4 + 9 + 16 + 25}} = \boxed{\frac{8}{\sqrt{54}}}$$

Problem 4 Exercise 18 p. 213

(6) I will use the notation of the book here (in class we used  $x \cdot y$  instead of  $x^T y$  for the standard scalar product in  $\mathbb{R}^n$ )

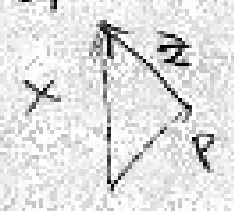
Prove that  $p^T(x-p) = 0$  or equivalently

$$p^T x = p^T p$$

$$p^T x = \left( \frac{x^T y}{y^T y} y \right)^T x = \left( \frac{x^T y}{y^T y} \right) y^T x = \frac{(x^T y)^2}{y^T y} \Rightarrow \text{Indeed } p^T x = p^T p$$

$$p^T p = \left( \frac{x^T y}{y^T y} y \right)^T \left( \frac{x^T y}{y^T y} y \right) = \frac{(x^T y)^2}{(y^T y)^2} y^T y = \frac{(x^T y)^2}{y^T y}$$

(6) Since  $p \perp z$  by Pythagoras Thm



$$\|x\|^2 = \|p\|^2 + \|z\|^2 = 6^2 + 8^2 = 36 + 64 = 100 \Rightarrow \|x\| = 10$$

Problem 5

(a) (i)  $\langle y, z \rangle = 1 \cdot 1 + 3 \cdot (-3) + 1 \cdot 1 + 1 \cdot 1 = 1 - 9 + 1 + 1 = -6$

(ii)  $\|y - z\| = \sqrt{(1-1)^2 + (3-(-3))^2 + (1-1)^2 + (1-1)^2} = 6$

(iii)  $\cos \theta = \frac{\langle y, z \rangle}{\|y\| \|z\|} = \frac{-6}{\sqrt{1+9+1+1} \sqrt{1+9+1+1}} = \frac{-6}{12} = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

(iv)  $\text{proj}_z y = \frac{\langle y, z \rangle}{\|z\|^2} z = \frac{-6}{12} (1, -3, 1, 1)^T = \left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^T$

(b) (i)  $\langle y, z \rangle = 1 \cdot 1 \cdot 3 + 3 \cdot (-3) \cdot 1 + 1 \cdot 1 \cdot 3 + 1 \cdot 1 \cdot 3 = 3 - 9 + 3 + 3 = 0$

(ii)  $\|y - z\| = \sqrt{(1-1)^2 \cdot 3 + (3-(-3))^2 \cdot 1 + (1-1)^2 \cdot 3 + (1-1)^2 \cdot 3} = 6$

(iii)  $\cos \theta = \frac{\langle y, z \rangle}{\|y\| \|z\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$  i.e.  $y \perp z$

(iv)  $\text{proj}_z y = \frac{\langle y, z \rangle}{\|z\|^2} z = 0$



3) (c) The answers are exactly the same as in the item (a)  
 (one says that there is an isomorphism between inner  
 product spaces  $\mathbb{R}^4$  and  $\mathbb{R}^{2 \times 2}$  given by

$$(x_1, x_2, x_3, x_4)^T \mapsto \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$$

This isomorphism send the vectors  $y$  and  $z$  of item (a)  
 to the corresponding matrices of the item (c)

(d) (i)  $\langle y, z \rangle = \int_0^1 e^{-2x} e^{2x} dx = \int_0^1 dx = 1$

(ii)  $\|y - z\|^2 = \int_0^1 (e^{-2x} - e^{2x})^2 dx = \int_0^1 (e^{4x} - 2 + e^{-4x}) dx =$

$$= \frac{1}{4} (e^{4x} - e^{-4x}) \Big|_0^1 - 2 = \frac{e^4 - e^{-4}}{4} - 2 = \frac{1}{2} \underbrace{\sinh 4}_{\text{hyperbolic sin}} - 2 =$$

$$\|y - z\| = \sqrt{\frac{e^4 - e^{-4}}{4} - 2} = \sqrt{\frac{1}{2} \sinh 4 - 2}$$

(iii)  $\cos \theta = \frac{\langle y, z \rangle}{\|y\| \|z\|}$

$$\|y\| = \sqrt{\int_0^1 e^{-4x} dx} = \sqrt{\left. -\frac{1}{4} e^{-4x} \right|_0^1} = \frac{\sqrt{1 - e^{-4}}}{2}$$

$$\|z\| = \sqrt{\int_0^1 e^{4x} dx} = \sqrt{\left. \frac{1}{4} e^{4x} \right|_0^1} = \frac{\sqrt{e^4 - 1}}{2}$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{\frac{(1 - e^{-4})}{4} (e^4 - 1)}} = \frac{1}{\sqrt{\frac{e^4 - 1 - 4e^{-4}}{4}}} = \frac{4}{e^2 - e^{-2}} = \frac{2}{\sinh 2}$$

$$\Rightarrow \theta = \arccos \frac{1}{e^2 - e^{-2}} = \arccos \frac{2}{\sinh 2}$$