PHY460/660: Homework 4

1. P3.2

$$f(x) = x^\nu.$$

(a)

$$\int_0^1 \mathrm{d}x\, |f(x)|^2 = \int_0^1 \mathrm{d}x\, x^{2\nu} = \frac{1}{2\nu+1} x^{2\nu+1}\Big|_0^1 < \infty \text{ if } 2\nu + 1 > 0.$$

Hence

$$\nu > -\frac{1}{2}.$$

If $\nu = -\frac{1}{2}$,

$$\int_0^1 \mathrm{d}x\, \frac{1}{x} = \ln x\Big|_0^1 \to \infty.$$

$$\int_{-\infty}^{\infty} \mathrm{d}x\, |\Psi(x,0)|^2$$

(b) If $\nu = \frac{1}{2}$,

$$\begin{aligned} f &= x^{1/2}:\ \text{yes}, \\ xf &= x^{3/2}:\ \text{yes}, \\ \frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{1}{2}x^{-1/2}:\ \text{no}. \end{aligned}$$

2. P3.4

(a)

$$\langle\phi|(H+K)|\psi\rangle = \langle\phi|H|\psi\rangle + \langle\phi|K|\psi\rangle = \langle H\phi|\psi\rangle + \langle K\phi|\psi\rangle = \langle (H+K)\phi|\psi\rangle,$$

i.e., $(H+K)$ is hermitian.

(b) Given $\langle\phi|Q|\psi\rangle = \langle Q\phi|\psi\rangle$
If

$$\langle\phi|\alpha Q|\psi\rangle = \langle \alpha Q\phi|\psi\rangle = \alpha^*\langle Q\phi|\psi\rangle = \alpha\langle\phi|Q|\psi\rangle,$$

hence

$$\alpha = \alpha^*.$$

(c)

$$\begin{aligned} \langle\phi|H|\psi\rangle &= \langle H\phi|\psi\rangle, \langle\phi|K|\psi\rangle = \langle K\phi|\psi\rangle, \\ \langle\phi|HK|\psi\rangle &= \langle H\phi|K|\psi\rangle = \langle KH\phi|\psi\rangle = \langle HK\phi|\psi\rangle. \end{aligned}$$

Hence,

$$[H, K] = 0.$$

(d)

$$\begin{aligned} \langle\phi|\hat{x}|\psi\rangle &= \int \mathrm{d}\,\phi^* x\psi = \int \mathrm{d}\,(x\phi)^* x\psi = \langle x\phi|\psi\rangle, \\ \langle\phi|H|\psi\rangle &= \langle\phi|(T+V)\psi\rangle, \\ \langle\phi|T|\psi\rangle &= -\frac{\hbar^2}{2m}\int \mathrm{d}x\,\phi^*\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} \propto \left[\phi^*\frac{\mathrm{d}\psi}{\mathrm{d}x}\right] - \int \mathrm{d}x\,\frac{\mathrm{d}\phi^*}{\mathrm{d}x}\frac{\mathrm{d}\psi}{\mathrm{d}x} \\ &= -\left[\psi\frac{\mathrm{d}\phi^*}{\mathrm{d}x}\right] + \int \mathrm{d}x\,\frac{\mathrm{d}^2\phi^*}{\mathrm{d}x^2}\psi = \langle T\phi|\psi\rangle. \end{aligned}$$

3. P.3.13

(a)

$$\begin{aligned} [AB, C] &= ABC - CAB = ABC - ACB + ACB - CAB \\ &= A[B, C] + [A, C]B. \end{aligned}$$

(b)

$$\begin{aligned} [x^n, p]\phi &= x^n p\phi - p(x^n\phi) = x^n p\phi - x^n p\phi + i\hbar n x^{n-1}\phi \\ &= i\hbar n x^{n-1}\phi. \end{aligned}$$

(c)

$$\begin{aligned} [f(x), p]\phi &= f(x)p\phi - p(f(x)\phi) = f(x)p\phi - f(x)p\phi - \phi(pf) \\ &= i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}\phi, \\ [f(x), p] &= i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}. \end{aligned}$$

4. P.3.27

$$\begin{aligned} \widehat{A}\psi_i &= a_i\psi_i, \widehat{B}\phi_i = b_i\phi_i, \\ \psi_1 &= \frac{1}{5}(3\phi_1 + 4\phi_2), \psi_2 = \frac{1}{5}(4\phi_1 - 3\phi_2). \end{aligned}$$

(a)

$$A = a_1, \Psi = \psi_1.$$

(b)

$$B : \psi_1 = \frac{1}{5}(3\phi_1 + 4\phi_2) = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2.$$

Possible results are: b_1 with probability $P_1 = \frac{9}{25}$; b_2 with probability $P_2 = \frac{16}{25}$.

(c) Note

$$\phi_1 = \frac{1}{5}(3\psi_1 + 4\psi_2), \phi_2 = \frac{1}{5}(4\psi_1 - 3\psi_2).$$

Measuring A after B:

- if $b_1, P_1 = \frac{9}{25}$,
- if $b_2, P_2 = \frac{16}{25}$.

Therefore,

$$P(a_1) = \frac{9}{25} \times \frac{9}{25} + \frac{16}{25} \times \frac{16}{25} = 0.54.$$