3/27/14

Exact functions

To determine exact equations use $f_{xy} = f_{yx}$

$f(x(T), y(T))$

$$f'(x(T), y(T)) = \frac{df}{dx}\frac{dx}{dT} + \frac{df}{dy}\frac{dy}{dT}$$

$$f'(x, y(x)) = \frac{df}{dx}\cdot\cancel{\frac{dx}{dx}} + \frac{df}{dy}\frac{dy}{dx} \quad \text{or} \quad f'(x) = f_x + f_y y'$$

Example

$f(x,y) = 3x^2y + 4xy + 3$

Partial of x $\rightarrow f_x = 6xy + 4y$ $\qquad f_{xy} = 6x+4 = f_{yx} = 6x+4$

$\rightarrow f_y = 3x^2 + 4x$ $\qquad f_{xy} = f_{yx} \leftarrow$ This statement is always True

Partial of y

$f'(x) = f_x + f_y y'$ $\leftarrow$ Must come out like chain rule Problems

$f(x, y(x)) = (6xy+4y) + (3x^2+4x)y'$

$\int (3x^2y + 4xy + 3)' = \int 0$

$3x^2y + 4xy\,3 = C$

$\Psi'(x,y) = 0$

Solve $\underbrace{(6xy+4y)}_{\Psi_x} + \underbrace{(3x^2+4x)}_{\Psi_y} y' = 0$

$\Psi_{xy} = 6x+4 = \Psi_{yx}$

Second option

$\int \Psi_y\,dy = \int 3x^2+4x\,dy = 3x^2y + 4xy + h(x)$

$\Psi_x = 6xy + 4y + h'(x) = 6xy + 4y$

$h'(x) = 0$

$h(x) = C$

$\boxed{\Psi(x,y) = 3x^2y + 4xy = C}$

$\Psi_x = 6xy + 4y$ $\qquad \Psi(x,y) = \int 6xy+4y\,dx = 3x^2y + 4xy + h(y)$

$\Psi_y = 3x^2 + 4x + h'(y)$

$\int h'(y) = \int 0$

$h(y) = C$

$\boxed{\begin{array}{c}\Psi(x,y) = 3x^2 + 4xy + C \\ \Psi(x,y) = C\end{array}}$

Example 2

$\Psi_{xy} = \Psi_{yx}$

$\underbrace{(3x^2y + y)}_{\Psi_x} + \underbrace{(x^3 + x + 1)}_{\Psi_y} y' = 0$

$\Psi_{xy} = 3x^2 + 1$

$\Psi_{yx} = 3x^2+1$

$\Psi(x,y) = \int \Psi_x\,dx = \int 3x^2y + y\,dx = x^3y + xy + h(y)$

$\frac{\Psi_y}{dy} = 3x + x + h'(y) = x^3 + x + 1$

$\int h'(y) = \int 1$

$h(y) = y + C$

Main Function

$\Psi(x,y) = x^3y + xy + y = C$

$\Psi(x,y) = \int \Psi_y\,dy = \int x^3 + x + 1 = x^3y + xy + y + h(x)$

$\frac{\Psi_x}{dx} = 3x^2y + y + h'(x) = 3x^2y + y$

$\int h'(x) = \int 0$

$h(x) = C$

$x^3y + xy + y + C = \Psi(x,y)$

Example 3

$$\underbrace{ye^x + 2y}_{\Psi_x} + \underbrace{(e^x + 2x + 3y)}_{\Psi_y} y' = 0$$

$\Psi_{xy} = e^x + 2$

$\Psi_{yx} = e^x + 2$

$\Psi(x,y) = \int \Psi_x dx = \int ye^x + 2y = ye^x + 2xy + h(y)$

$\Psi_y = e^x + 2x + h'(y) = e^x + 2x + 3y$

$\int h'(y) = \int 3y$

$h(y) = 3/2\, y^2 + C$ $\quad \boxed{ye^x + 2xy + 3/2\, y^2 = C = \Psi(x,y)}$

$\Psi(x,y) = \int \Psi_y dy = \int e^x + 2x + 3y\, dy = ye^x + 2yx + 3/2\, y^2 + h(x)$ $\quad \|$

$\Psi_x = ye^x + 2y + h'(x) = ye^x + 2y$

with intial value of $y_0 = 1$

$\int h'(x) = \int 0$ $\quad \boxed{\Psi(x,y) = ye^x + 2yx + 3/2\, y^2 + C}$

$h(x) = C$

$2(1)e^0 + 4(0)(1) + 3(1)^2 = C$

$2 + 3 = C \qquad C = 5 \longrightarrow \Psi(x,y) = ye^x + 2yx + 3/2\, y^2 = 5$

Are All seperable problems Exact?

$$y' = \frac{f(x)}{f(y)} \rightarrow f(y)y' = f(x) \rightarrow \underbrace{-f(x)}_{\Psi_x} + \underbrace{f(y)}_{\Psi_y} y' = 0 \qquad \Psi_{xy} = 0 = \Psi_{yx}$$

So yes they are all Exact!

$y' + 3y = 7$

Way ① $\quad \mu(x) = e^{\int 3dx} = e^{3x}$

$e^{3x}y' + e^{3x}3y = 7e^{3x} \qquad \int (e^{3x}y)' = \int 7e^{3x} \qquad e^{3x}y = 7/3\, e^{3x} + C$

$\boxed{y = 7/3 + \frac{C}{e^{3x}}}$

Way ② $\quad y' = 7 - 3y \qquad y' = -3(-7/3 + y) \qquad \int \frac{y'}{(-7/3 + y)} = \int -3 \qquad \ln(-7/3 + y) = -3x + C$

$-7/3 + y = Ce^{-3x} \qquad y = Ce^{-3x} + 7/3$

Way ③ $\quad 3 + \left(\frac{1}{y - 7/3}\right) y' = 0$

Solve for HW