9/22/14 MATH 2850

- Preview of chain rule:

$\mathbb{R}^1 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^1$

$g(t) = \begin{bmatrix} t^3-3t \\ t^2+4 \end{bmatrix}$ $\quad f(x,y) = x^2y + y^2$

$(f \circ g)'(3)$ $\quad g'(t) = \begin{bmatrix} 3t^2-3 \\ 2t \end{bmatrix}$ $\quad f'(x,y) = \begin{bmatrix} 2xy & x^2+2y \end{bmatrix}$

$(f' \circ g)(3)g'(3)$ $\quad f'(18,13)\ g'(3)$

$= \begin{bmatrix} 2(18)(13) & (18)^2+2(13) \end{bmatrix} \begin{bmatrix} 24 \\ 6 \end{bmatrix}$

$= 2(18)(13)(24) + ((18)^2 + 2(13))(6)$

- $x^3 + xy^2 + z^4 = 17$

Consider x as a function of y and z

Compute $\frac{dx}{dy}$ $\quad 3x^2\frac{dx}{dy} + \frac{dx}{dy}y^2 + x2y + 0 = 0$

$\left(\frac{dx}{dy}\right)(3x^2+y^2) = -2xy$ $\quad \frac{dx}{dy} = \frac{-2xy}{3x^2+y^2}$

4.4 & 14.5

- Linear approximations
- Tangent spaces
- Directional Derivatives
- Remark: If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $P \in \mathbb{R}^n$ then the derivative $f'(P)$ defines a linear function

$L: \mathbb{R}^n \to \mathbb{R}^m$

$Q \longmapsto f'(P)Q$

L, linear means:

① $L(Q+R) = L(Q) + L(R)$

② $L(Q\alpha) = L(Q)\alpha$ $\quad$ α is a number

ex: $F: \mathbb{R}^1 \to \mathbb{R}^1$ $\quad$ *calc 1

$F(x) = x^3$; $\quad P = 2 \quad F'(x) = 3x^2 \quad F'(2) = 12$

$L: \mathbb{R}^1 \to \mathbb{R}^1$

$Q \mapsto 12Q$

$Q = 5 \quad R = 8 \quad L(Q+R) = 12(5+8) = 12(13) = 156$ ✓

$L(Q) + L(R) = 12(5) + 12(8) = 60 + 96 = 156$ ✓

$\alpha = 4 \quad L(Q\alpha) = 12(5 \cdot 4) = 12(20) = 240$ ✓

$L(Q)\alpha = (12 \cdot 5) \cdot 4 = 60 \cdot 4 = 240$ ✓

$g: \mathbb{R}^2 \to \mathbb{R}^3$

$$g(u,v) = \begin{bmatrix} u^2v \\ uv \\ u-v \end{bmatrix} \quad P = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad g' \begin{bmatrix} 2uv & u^2 \\ v & u \\ 1 & -1 \end{bmatrix}$$

$$g(1,2) \begin{bmatrix} 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \quad Q = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad R = \begin{bmatrix} 8 \\ 7 \end{bmatrix} \quad \alpha = 4$$

$$L(Q+R) = g'(1,2)\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 13 \\ 9 \end{bmatrix} = \begin{bmatrix} 4(13) + 1(9) \\ 2(13) + 1(9) \\ 1(13) - 1(9) \end{bmatrix} = \begin{bmatrix} 61 \\ 35 \\ 4 \end{bmatrix}$$

$$L(Q) + L(R) = \begin{bmatrix} 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} = \begin{bmatrix} 4(5) + 2 \\ 2(5) + 2 \\ 1(5) - 2 \end{bmatrix} + \begin{bmatrix} 4(8) + 7 \\ 2(8) + 7 \\ 8 - 7 \end{bmatrix} = \begin{bmatrix} 61 \\ 35 \\ 4 \end{bmatrix}$$

$L(Q\alpha) = L(Q)\alpha$

Linear approximations

Given $f: \mathbb{R}^n \to \mathbb{R}^m$ and $P \in \mathbb{R}^n$ and $\Delta P \in \mathbb{R}^n$

$$f(P+\Delta P) \approx f(P) + f'(P)\Delta P \underbrace{\qquad\qquad}_{\text{error}}$$

know $\dfrac{|E_P(P,\Delta P)|}{|\Delta P|} \to 0$ as $(\Delta P) \to 0$

- ex: $\mathbb{R}^1 \to \mathbb{R}^1$

  $x \mapsto x^3$

  $f(x) = x^3$ $P = 2$ $f'(2) = 12$

  Set $\Delta P = 0.03$

$$f(P+\Delta P) = f(2.03) \approx f(2) + f'(2)(0.03)$$
$$= 8 + 12(0.03)$$
$$= 8.36$$

- ex: $\mathbb{R}^2 \to \mathbb{R}^3$

$$(u,v) \mapsto \begin{bmatrix} u^2v \\ uv \\ u-v \end{bmatrix} \quad P = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad g(1,2) = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$g'(1,2) = \begin{bmatrix} 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Delta P = \begin{bmatrix} 0.02 \\ -0.03 \end{bmatrix}$$

$$g(1.02, 1.97) \approx \overset{f(P)}{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} + \overset{f'(P)}{\begin{bmatrix} 4 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}} \overset{\Delta P}{\begin{bmatrix} 0.02 \\ -0.03 \end{bmatrix}} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0.08 - 0.03 \\ 0.04 - 0.03 \\ 0.02 + 0.03 \end{bmatrix}$$

$$= \begin{bmatrix} 2.05 \\ 2.01 \\ -0.95 \end{bmatrix}$$