

Recitation on induction (Solutions)

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1. Use mathematical induction to prove that 3 divides $n^3 + 2n$ whenever n is a nonnegative integer.

We are going to prove by induction on $n \geq 0$, that $n^3 + 2n$ is divisible by 3.

Basis. $n = 0$. $0^3 + 0 = 0$ is divisible by 3, so the proposition holds for $n = 0$.

IH. Assume that the proposition is true for $n = k$, where k is some integer, $k \geq 0$. In other words we assume that $k^3 + 2k$ is divisible by 3, i.e. $k^3 + 2k = 3p$, p is an integer.

IS. We need to prove that $(k+1)^3 + 2(k+1)$ is divisible by 3.

$$\begin{aligned} (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 = \\ &= (k^3 + 2k) + 3(k^2 + k + 1) \\ &= 3p + 3(k^2 + k + 1) \quad \text{by IH} \end{aligned}$$

Since both terms have factor 3, the sum is divisible by 3. So, induction step is proved. By Induction Principle the proposition is true for any n .

2. Use induction to prove that

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$$

We are going to prove by induction on $n \geq 0$ that the equality is true.

Basis. $n = 0$. LHS = 3, RHS = $3(5^{0+1} - 1)/4 = 3$. So, LHS = RHS.

IH. Assume that the proposition is true for $n = k$, where k is some integer $k \geq 0$. In other words we assume that for some $k \geq 0$ $3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k = 3(5^{k+1} - 1)/4$

IS. We need to prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k + 3 \cdot 5^{k+1} = 3(5^{(k+1)+1} - 1)/4$

$$\begin{aligned} 3 + 3 \cdot 5 + 3 \cdot 5^2 + \dots + 3 \cdot 5^k + 3 \cdot 5^{k+1} &= 3(5^{k+1} - 1)/4 + 3 \cdot 5^{k+1} \quad \text{(by IH)} \\ &= 3(5^{k+1} - 1)/4 + 3 \cdot 4 \cdot 5^{k+1}/4 = 3(5 \cdot 5^{k+1} - 1)/4 = 3(5^{(k+1)+1} - 1)/4 \quad \text{(algebra)} \end{aligned}$$

So, IS is proved and by Induction principle the formula is proved for any $n \geq 0$.

3. Prove that for any $n > 1$

$$1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

We are going to prove it by induction on $n \geq 2$.

Basis. $n = 2$, LHS = $1 + 1/4 = 5/4$, RHS = $2 - (1/2) = 3/2$, LHS < RHS.

IH. Assume that the inequality holds for $n = k$, where k is some integer $k \geq 2$, i.e. we

assume that for some $k \geq 2$ $1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k}$

IS. We need to prove that $1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

$$\text{LHS} = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}, \text{ by IH.}$$

To prove inductive step it is sufficient to show that $2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$

We can note that $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)} > \frac{1}{(k+1)^2}$, since $k < k+1$.

By rearranging the terms we can get what we need, $\frac{1}{(k+1)^2} - \frac{1}{k} < -\frac{1}{k+1}$

In this way the inductive step is proved. By induction principle the inequality is true for any $n > 1$.

4. Harmonic numbers H_k , $k = 1, 2, 3, \dots$ are defined by

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}.$$

Use mathematical induction to prove that $H_{2^n} \geq 1 + \frac{n}{2}$

Basis. $n = 1$, LHS = $H_{2^1} = H_2 = 1 + \frac{1}{2} = \frac{3}{2}$, RHS = $1 + \frac{1}{2} = \frac{3}{2}$ and the proposition $\frac{3}{2} \geq \frac{3}{2}$ is true.

IH. Assume that for $n = k$, where k is some integer $k \geq 1$, the inequality is true, i.e.

$$H_{2^k} \geq 1 + \frac{k}{2}$$

IS. We need to prove that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$. By the given definition of harmonic function

$$\text{we have } H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} = H_{2^k} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}}$$

$$\text{By IH we have } H_{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}}$$

To prove inductive step it is sufficient to show that $\frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} \geq \frac{1}{2}$.

In the left hand side we have the sum of $2^{k+1} - 2^k = 2^k(2 - 1) = 2^k$ terms and each term

$$\frac{1}{2^k + s} \geq \frac{1}{2^k + 2^k} = \frac{1}{2^{k+1}},$$

$$\text{so that the total sum is } \frac{1}{2^k+1} + \dots + \frac{1}{2^{k+1}} \geq 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}.$$

We proved inductive step, and by induction principle the inequality is true for all integer $n \geq 1$.

5. Prove by induction that for any integers a and b $(a \pm b) \mid (a^n \pm b^n)$, where n is a positive integer, $n \geq 1$.

We are going to prove by induction on $n \geq 1$, that for any integers a and b , $(a^n \pm b^n)$ is divisible by $(a \pm b)$. By the definition of divisibility it is sufficient to prove that for any $n \geq 1$, for any a and b there exists an integer m such that $(a^n \pm b^n) = m(a \pm b)$.

Basis. $n = 1$, $(a \pm b) = 1(a \pm b)$.

IH. Assume that proposition is true for $n = k$, where k is some integer, $k \geq 1$. In other words, we assume that for some $k \geq 1$ and any a and b there exists an integer m such that $(a^k \pm b^k) = m(a \pm b)$.

IS. We need to prove that the proposition holds for $n = k + 1$, i.e. for any a, b there exists some integer s , such that $(a^{k+1} \pm b^{k+1}) = s(a \pm b)$

To reduce $k+1$ case to the case $n=k$, we consider the following identity:

$$a^{k+1} - b^{k+1} = a(a^k - b^k) + a b^k - b^{k+1} = a(a^k - b^k) + b^k (a - b)$$

By IH we have for these a and b some integer m such that $(a^k - b^k) = m(a - b)$, so:

$$a(a^k - b^k) + b^k (a - b) = a m(a - b) + b^k (a - b)$$

So, $a^{k+1} - b^{k+1} = (a - b)(a m + b^k) = (a - b) s$, where $s = a m + b^k$.

We proved IS, and by Induction Principle the proposition is true for any $n \geq 1$.

6. A sequence of numbers a_n is defined recursively as follows:

$$a_0 = 0;$$

$$a_{n+1} = 2a_n + 1, \text{ for } n \geq 0$$

Prove that $a_n = 2^n - 1$.

Proof by induction on $n \geq 0$, that the recursively defined sequence satisfies the closed formula $a_n = 2^n - 1$.

Basis. $n=0$, $a_0=0$, that agrees with the formula $a_0 = 2^0 - 1 = 1 - 1 = 0$.

IH. Assume that for $n = k$, k is an integer $k \geq 0$, that the formula is correct, i. e. $a_k = 2^k - 1$.

IS. We need to prove the formula for $n = k+1$, i.e. that $a_{k+1} = 2^{k+1} - 1$.

$$a_{k+1} = 2a_k + 1 \text{ (by recursive definition)}$$

$$= 2(2^k - 1) + 1 \text{ (by IH)}$$

$$= 2^{k+1} - 2 + 1 = 2^{k+1} - 1$$